Physics Letters A 382 (2018) 1092–1099

Contents lists available at ScienceDirect

Physics Letters A

www.elsevier.com/locate/pla

The stationary sine-Gordon equation on metric graphs: Exact analytical solutions for simple topologies

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ARTICLE INFO

Article history:
Received 27 September 2016
Received in revised form 13 December 2017
Accepted 22 December 2017
Available online 27 February 2018
Communicated by A.P. Fordy

Keywords:
The stationary sine-Gordon equation
Metric graphs
The star graph
The Gibbs free-energy functional
Jacobi's elliptic functions

ABSTRACT

We consider the stationary sine-Gordon equation on metric graphs with simple topologies. Exact analytical solutions are obtained for different vertex boundary conditions. It is shown that the method can be extended for tree and other simple graph topologies. Applications of the obtained results to branched planar Josephson junctions and Josephson junctions with tricrystal boundaries are discussed.

1. Introduction

Nonlinear wave equations have numerous applications in different topics of physics and natural sciences (see, e.g., [1–6]). Recently they have attracted a lot attentions in the context of soliton transports in networks and branched structures [7–19]. Wave dynamics in networks can be modelled by nonlinear evolution equations on metric graphs. This fact greatly facilitates the study of soliton transports in branched systems. Metric graph is a system of bonds which are assigned a length and connected at the vertices according to a rule, called "topology of a graph". Solitons and other nonlinear waves in branched systems appear in different systems of condensed matter, polymers, optics, neuroscience, DNA and many others.

In condensed matters, very important branched systems where solitons can appear are the Josephson junction networks [20]–[21]. The phase difference in a Josephson junction obeys the sine-Gordon equation [22]. Josephson junction networks can therefore be effectively modelled by the sine-Gordon equation on metric graphs. The early treatment of superconductor networks consisting of Josephson junctions meeting at one point dated back to [23,24]. An interesting realization of Josephson junction networks at tricrystal boundaries was discussed earlier in [25], which inspired later detailed study of the problem using the sine-Gordon equation on networks in [17,26,27]. Discrete sine-Gordon equations were also used in [20,21,28] to describe different networks of Josepshon junctions having several junctions on each wire of a network. Recently, a 2D sine-Gordon equation on networks was studied by considering Y and T junctions [18]. Discrete sine-Gordon equations on networks were also considered in [29].

In this paper we address the problem of stationary sine-Gordon equations on metric graphs by focusing on exact analytical solutions for simple graph topologies. Such a one-dimensional, stationary sine-Gordon equation describes, for instance, the transverse component of the phase difference in a 2D Josephson junction in a constant magnetic field. The derivative of the phase difference presents the local magnetic field in the system [30–32]. Planar Josephson junctions were studied in [31,32] on the basis of solutions of the stationary sine-Gordon equation on a finite interval. Here, we use a similar approach to solve the stationary sine-Gordon equation on metric graphs. Two types of vertex boundary conditions are considered providing different conservation laws, continuity of the wave function and its derivatives. The model proposed in this work can be used to describe static solitons in 2D Josephson junctions interacting with constant magnetic field [31,32]. The results are then extended for metric tree graphs consisting of finite bonds. The study can be generalized to other simple

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https://doi.org/10.1016/j.physleta.2017.12.057
0375-9601/

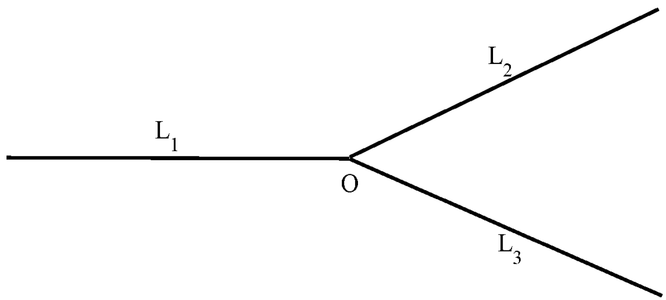

Fig. 1. Sketch of a metric star graph. L_j is the length of the jth bond with $j = 1, 2, 3$.

graph topologies, which can be constructed using star and loop graphs.

This paper is organized as follows. In the next section we give a formulation of the problem by deriving the boundary conditions for the a star graph and provide the exact analytical solutions for special cases. In Section 3, we extend the computational procedures to metric tree graphs. In Section 4, we explore the stability of the obtained solutions. Finally, Section 5 presents some concluding remarks.

2. Vertex boundary conditions and exact solutions for star graph

The static sine-Gordon equation on a metric graph presented in Fig. 1 can be written as

$$\frac{d^2}{dx^2}\phi_j = \frac{1}{\lambda_j^2}\sin(\phi_j),\ 0 < x < L_j, \tag{1}$$

where the wave function ϕ_j is assigned to each bond of the graph and $j = 1, 2, 3$ is the bond number. For wave equations on networks, the connections of the network wires at the vertices are provided by the vertex boundary conditions. In case of linear wave equations, the underlying constraint to derive vertex boundary conditions is the self-adjointness of the problem [33,34]. However, for nonlinear case one should use different conservation laws [7,9,18]. Here, for the stationary sine-Gordon equation we impose two types of boundary conditions. The first type provides the continuity of the 'weighted' wave function derivatives

$$\lambda_1 \left.\frac{d\phi_1}{dx}\right|_{x=0} = \lambda_2 \left.\frac{d\phi_2}{dx}\right|_{x=0} = \lambda_3 \left.\frac{d\phi_3}{dx}\right|_{x=0} \tag{2}$$

and conservation of the magnetic self-field flux at the vertex, which are given as

$$\lambda_1\phi_1|_{x=0} + \lambda_2\phi_2|_{x=0} + \lambda_3\phi_3|_{x=0} = 0. \tag{3}$$

The second type of vertex conditions has the form of wave function continuity at the vertex

$$\phi_1|_{x=0} = \phi_2|_{x=0} = \phi_3|_{x=0}, \tag{4}$$

and conservation of the applied magnetic flux at the vertex, which is given as

$$\lambda_1^2 \left.\frac{d\phi_1}{dx}\right|_{x=0} + \lambda_2^2 \left.\frac{d\phi_2}{dx}\right|_{x=0} + \lambda_3^2 \left.\frac{d\phi_3}{dx}\right|_{x=0} = 0. \tag{5}$$

In terms of Josephson junction networks the vertex conditions (3) and (5) imply Kirchhoff rules for self-induced and external magnetic field fluxes. The boundary condition at the end of each bond is given by

$$\left.\frac{d\phi_j}{dx}\right|_{x=L_j} = 2H_j, \tag{6}$$

with H_j being the homogeneous external magnetic field applied along the jth bond. Such boundary conditions may appear in, e.g., branched graphene nanoribbons [35,36]. Since strained graphene creates pseudo magnetic field [37], in this way it is possible to obtain magnetic fields of different strengths along the different junctions when they have different strains.

The boundary conditions (2)–(3) are consistent with other models of Josephson junction networks previously studied theoretically in [17,25,31,32] as well as experimentally in [38–40]. Detailed derivation of these boundary conditions are resented in Appendix A. Exact solutions of Eq. (1) have been obtained earlier in [31,32,41] for the second boundary conditions on a finite interval. Here, we use an approach similar to that in [31,32] to obtain exact analytical solutions of Eq. (1) for the boundary conditions (2)–(6).

2.1. Solution of type I

Our purpose is to obtain exact analytical solutions of the problem given by Eqs. (1)–(6). A solution of Eq. (1) without boundary conditions can be written as [31,32]

$$\phi_j^{(\pm)}(x) = (2n_j+1)\pi \pm 2\arcsin\left\{k_j \,\mathrm{sn}\left[\frac{x - x_{0,j}^{(\pm)}}{\lambda_j}, k_j\right]\right\} \tag{7}$$

where k_j and $x_{0,j}^{(\pm)}$ are integration constants and sn is Jacobi's elliptic function. Depending on the value of k_j that is determined by the constraint $|H_j\lambda_j| \le |k_j| \le 1$, we refer to the solution as solution of type 1 [31]. Taking into account that

$$\frac{d\phi_j^{(\pm)}}{dx} = \pm\frac{2k_j}{\lambda_j}\,\mathrm{cn}\left[\frac{x - x_{0,j}^{(\pm)}}{\lambda_j}, k_j\right], \tag{8}$$

from boundary condition (6) we have

$$x_{0,j}^{(\pm)} = L_j - \lambda_j F\left[\arccos\left(\pm\frac{H_j\lambda_j}{k_j}\right), k_j\right]. \tag{9}$$

Here, cn is Jacobi's elliptic function [42] and $F(\varphi, k)$ is the elliptic integral of the first kind [42]. Then solution of type 1 of the sine-Gordon equation on a metric star graph with the boundary conditions (2)–(3) can be written as

$$\phi_j^{(\pm)}(x) = (2n_j+1)\pi \pm$$
$$\pm 2\arcsin\left\{k_j\,\mathrm{sn}\left[\frac{x - L_j}{\lambda_j} + F\left[\arccos\left(\pm\frac{H_j\lambda_j}{k_j}\right), k_j\right], k_j\right]\right\}.$$

The vertex boundary conditions (2) and (3) lead to the following system of transcendental equations for finding k_j:

$$\sum_{j=1}^{3}\lambda_j \arcsin\left\{k_j\,\mathrm{sn}\left[\frac{L_j}{\lambda_j} - F\left[\arccos\left(\pm\frac{H_j\lambda_j}{k_j}\right), k_j\right], k_j\right]\right\}$$
$$= \pm\frac{1}{2}\sum_{j=1}^{3}(2n_j+1)\pi\lambda_j, \tag{10}$$

$$k_1\,\mathrm{cn}\left[\frac{L_1}{\lambda_1} - F\left[\arccos\left(\pm\frac{H_1\lambda_1}{k_1}\right), k_1\right], k_1\right] =$$
$$= k_2\,\mathrm{cn}\left[\frac{L_2}{\lambda_2} - F\left[\arccos\left(\pm\frac{H_2\lambda_2}{k_2}\right), k_2\right], k_2\right], \tag{11}$$

$$k_1\,\mathrm{cn}\left[\frac{L_1}{\lambda_1} - F\left[\arccos\left(\pm\frac{H_1\lambda_1}{k_1}\right), k_1\right], k_1\right] =$$
$$= k_3\,\mathrm{cn}\left[\frac{L_3}{\lambda_3} - F\left[\arccos\left(\pm\frac{H_3\lambda_3}{k_3}\right), k_3\right], k_3\right]. \tag{12}$$