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Sub- and super-diffusion on Cantor sets: Beyond the paradox

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ABSTRACT

There is no way to build a nontrivial Markov process having continuous trajectories on a totally disconnected fractal embedded in the Euclidean space. Accordingly, in order to delineate the diffusion process on the totally disconnected fractal, one needs to relax the continuum requirement. Consequently, a diffusion process depends on how the continuum requirement is handled. This explains the emergence of different types of anomalous diffusion on the same totally disconnected set. In this regard, we argue that the number of effective spatial degrees of freedom of a random walker on the totally disconnected Cantor set is equal to $n_{sp} = [D] + 1$, where $[D]$ is the integer part of the Hausdorff dimension of the Cantor set. Conversely, the number of effective dynamical degrees of freedom (d_s) depends on the definition of a Markov process on the totally disconnected Cantor set embedded in the Euclidean space E^n ($n \geq n_{sp}$). This allows us to deduce the equation of diffusion by employing the local differential operators on the F^α-support. The exact solutions of this equation are obtained on the middle-ϵ Cantor sets for different kinds of the Markovian random processes. The relation of our findings to physical phenomena observed in complex systems is highlighted.

1. Introduction

Diffusion phenomena play an important role in physics, chemistry, technological processes, and biology [1–4]. Classical diffusion is associated with a Brownian motion. The laws of Brownian motion crucially rely on the hypothesis that the steps of a random walker are small (with finite variance) and uncorrelated. Whenever these assumptions are violated, the standard diffusion picture breaks down, leading to emergence of an anomalous diffusion [5]. In contrast to the Gaussian diffusion, which is characterized by the linear dependence of the mean square displacement on the time, the anomalous diffusion is non-universal in that it involves a parameter called the fractal dimension of a random walk [2]. Accordingly, for long enough times the mean square displacement of an ensemble of walkers behaves asymptotically as

$$< r^2 >= K_\gamma t^\gamma, \quad (1)$$

where K_γ is the generalized diffusion coefficient and $\gamma = 2/D_w$ is the diffusion exponent. In the case of normal diffusion $\gamma = 1$ whereas the sub- and super-diffusion are characterized by $\gamma < 1$ and $\gamma > 1$ respectively [1–5]. The generalized Einstein law provides the relation between the fractal dimension of random walk D_w, electric resistance exponent ζ and spectral dimension of medium d_s, which reads as $D_w = 2\zeta/(2 - d_s)$ [6]. Furthermore, many types of fractals (but not all) obey the Alexander–Orbach relation

$$D_w = \frac{2D}{d_s}, \quad (2)$$

where D is the fractal (e.g. Hausdorff, box-counting, or self-similarity) dimension, such that $\zeta = D_w - D$ [5,6]. For path-connected fractals $d_s \leq D$ and so $\gamma \leq 1$ while $D_w \geq 2$ [7]. Accordingly, the random walk on path-connected fractals has become a paradigmatic model for the sub-diffusion observed in a great variety of physical systems (see Refs. [1–9] and references therein). On the other hand, the super-diffusion is commonly associated with Lévy flights [9,10]. At the same time, characteristic features of many physical systems can be modeled using totally disconnected fractals, e.g. Cantor sets embedded in E^n [11–14]. The ternary Cantor set of real numbers was introduced by Georg Cantor in order to illustrate the statement that a perfect set does not need to be everywhere dense. Geometrically, it is constructed by iterative deletion of open middle-third intervals from remaining intervals of the previous iteration, starting from the unit interval $[0, 1]$ ad infinitum. The ternary Cantor set is a self-similar, totally discon-

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nected, bounded, compact, perfect set which is nowhere dense on [0, 1]. Its Hausdorff dimension is equal to the similarity dimension $D = \ln 2/\ln 3$. The middle-ϵ Cantor sets with the Hausdorff dimension

$$D = \frac{\ln 2}{\ln 2 - \ln(1-\epsilon)} \tag{3}$$

can be constructed by the iterative deletion of open intervals of length equal to the ϵ fraction of the length of intervals remaining after the previous iteration [15]. The Cantor set in E^n (also called the Cantor dust) with the Hausdorff dimension $0 < D < n$ can be constructed either by similar iterative method, starting from n-dimensional unit cube $[0, 1]^n$, or as the Cartesian product of n orthogonal Cantor sets of the same or different Hausdorff dimensions. Every Cantor set (dust) is homeomorphic to the ternary Cantor set [16]. Furthermore, it has been proved (see Ref. [17]) that any compact metric space is a continuous image of the ternary Cantor set. Due to these remarkable properties the totally disconnected Cantor sets have found a celebrated place in mathematical analysis and its applications. In particular, stochastic processes on Cantor sets are an active topic of research [18–31]. In this regard, the ternary Cantor set is interesting because the diffusion on it exhibits some apparent paradoxes. Specifically, in some works, (see Refs. [18–24]) it was suggested that the Markov processes defined on the totally disconnected Cantor set lead to the super-diffusion allied with the set's self-similarity. Conversely, it was argued (see [28–31]) that the ultrametric properties of the totally disconnected Cantor sets stipulate the sub-diffusive nature of the Markov processes. It has been also recognized that local and average measurements can display different asymptotic behavior [12,33]. Accordingly, alternative models of the anomalous diffusion on the ternary Cantor set were suggested and disputed in the literature (see, for example, Refs. [32–36] and references therein). In this work, we argue that diffusion on the totally disconnected Cantor set can be either the super-, or the sub-diffusion, depending on how the random walk is constructed. In this regard, we stress that the number of effective dynamical degree of freedom of a random walker on the Cantor set depends on the Markov process definition, while the number of effective spatial degree of freedom is equal to the minimum dimension of the Euclidean space in which the Cantor set can be bi-Lipschitz embedded. This explains the emergence of different types of the anomalous diffusion on the ternary Cantor set. The diffusion equation describing different types of diffusion is deduced and solved for the middle-ϵ Cantor sets. The physical implications of these findings are discussed. The rest of the paper is organized as follows. Sec. 2 is devoted to the analysis of different Markov processes on the totally disconnected Cantor set. The processes associated with the sub- and super diffusion on the ternary Cantor set are outlined. The predicted values for the diffusion exponents are compared with available data from numerical simulations of Lévy flights and walks on the Cantor sets. In Sec. 3 the equation of diffusion is deduced using the algorithmic F^α-calculus based on the Riemannian-like method. The relations between the orders of F^α-derivatives and the numbers of the effective degrees of freedom are established. The probability density distributions functions are found for different types of diffusion on the middle-ϵ Cantor sets. Some physical phenomena associated with different types of anomalous diffusion on the Cantor sets are highlighted. The main findings and conclusions are outlined in Sec. 4.

2. Markov processes on Cantor sets

It is a straightforward matter to understand that there is no way to build a nontrivial Markov process having continuous trajectories on a totally disconnected Cantor set embedded in the Euclidean space E^n. Therefore, in order to delineate the diffusion process, one needs to relax the continuum requirement. In this way, different properties of the ternary Cantor set were explored to define the Markov random processes on it. Specifically, the Brownian motion on the totally disconnected fractal set was introduced by Fujita [18]. This model was used to determine the spectral dimension of the Cantor set from the growth order of eigenvalues of the Brownian motion generator. Further, the transition probability densities for generalized one-dimensional diffusion processes were studied in [19]. Later, Takahashi and Tamura [20] have defined diffusion processes on totally disconnected self-similar fractal sets as the limits of suitably scaled random walk. On the other hand, Evans [21] has introduced Markov processes with stationary independent increments taking values in a non-discrete, locally compact, metrizable, totally disconnected Abelian Cantor group. Further, Aldous and Evans [22] have used the Dirichlet form methods to construct and analyze a general class of reversible Markov processes with totally disconnected state space. At the same time, Freiberg [23] has introduced the measure theoretic Dirichlet forms on compact subsets of the real line. Using the technique of Dirichlet–Neumann-bracketing, he has obtained the estimates of the eigenvalue counting functions of the associated measure geometric Laplacians. Using the Dirichlet form technique, Karwowski [24] has developed a model of diffusion on the real line with jumps on the Cantor set which preserve the ultrametric feature of random process on unit ball of 2-adic numbers. In this regard, it is pertinent to note that the Markov processes studied in Refs. [18–24] lead to the super-diffusion on the ternary Cantor set.

Alternatively, Lobus [25] has constructed a strong Markov processes on Cantor sets from the Wiener processes by means of time change, killing, and space transformation. Bhamidi et al. [26] have proposed an analogue of Brownian motion that has as its state space an arbitrary closed subset of the line that is unbounded above and below. It was shown that there is a unique such process, which turns out to be automatically a reversible Feller–Dynkin Markov process. This process is martingale and has the identity function as its quadratic variation process. So, it is continuous in the sense that its sample paths don't skip over points. It was also recognized that the totally disconnected Cantor sets exhibit a natural ultrametric structure [27,28]. An analogue of the Riemannian geometry for an ultrametric Cantor set was suggested in Ref. [29]. Consequently, Pearson and Bellissard [29] have defined a probability measure, Dirichlet forms, and associated analogue of the Laplace–Beltrami operator. In this way, it was deduced that the diffusion on the ultrametric Cantor set exhibits nonstandard sub-diffusive behavior, as $t \to 0$. Later, Bakhtin [30] has defined an analog of Brownian motion on the triadic Cantor set by introducing requirements on the Markov semigroup, most important of which are isometry invariance, and scale invariance. This allows to build up the jump statistics, while the generators of the symmetric self-similar Markov process play the role of the Laplacian on the Cantor set. More recently, Kigami [31] has shown that a transient random walk on a tree induces a Dirichlet form on its Martin boundary, which is the Cantor set. This was used to define an intrinsic metric on the Cantor set associated with the random walk. Accordingly, the harmonic measure and induced Dirichlet form on the Cantor set were explicitly expressed in terms of the effective resistances. Consequently, the asymptotic behaviors of the heat kernel, the jump kernel, and moments of displacements were obtained. In this regard, it is pertinent to point out that, in contrast to Markov processes studied in Refs. [18–24], the Markov processes defined in Refs. [25–31] lead to the sub-diffusion on the ternary Cantor set.

In this background, we stress that, generally, the type of diffusion is determined by the numbers of effective dynamical and spatial degrees of freedom of a random walker (see Refs. [37,38]).

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