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Roughness as classicality indicator of a quantum state

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ABSTRACT

We define a new quantifier of classicality for a quantum state, the Roughness, which is given by the $\mathcal{L}^2(\mathbb{R}^2)$ distance between Wigner and Husimi functions. We show that the Roughness is bounded and therefore it is a useful tool for comparison between different quantum states for single bosonic systems. The state classification via the Roughness is not binary, but rather it is continuous in the interval $[0, 1]$, being the state more classic as the Roughness approaches to zero, and more quantum when it is closer to the unity. The Roughness is maximum for Fock states when its number of photons is arbitrarily large, and also for squeezed states at the maximum compression limit. On the other hand, the Roughness approaches its minimum value for thermal states at infinite temperature and, more generally, for infinite entropy states. The Roughness of a coherent state is slightly below one half, so we may say that it is more a classical state than a quantum one. Another important result is that the Roughness performs well for discriminating both pure and mixed states. Since the Roughness measures the inherent quantumness of a state, we propose another function, the Dynamic Distance Measure (DDM), which is suitable for measure how much quantum is a dynamics. Using DDM, we studied the quartic oscillator, and we observed that there is a certain complementarity between dynamics and state, i.e. when dynamics becomes more quantum, the Roughness of the state decreases, while the Roughness grows as the dynamics becomes less quantum.

1. Introduction

To determine if the system is classical or quantum is one of the most intriguing physics questions of the last decades. The first challenging question was to measure the quantum state. Much effort in this direction was made by several researchers, with many advances, both theoretical [1,2] and experimental [3–6]. The first approach to this problem was based on Ehrenfest theorem [7–16] which states that, under certain conditions, the centroid of a wave-packet state will follow a classical trajectory. Zurek and Paz [17,18] argue that the quantum system is never isolated, and thus the dynamics of a macroscopic object is modified by the surrounding objects that interact with it. This is the *Decoherence Approach to Classical Limit of Quantum Mechanics* [17–25]. Up to our knowledge, Ballentine and collaborators [26–30] where the first to address the question of which classical dynamics would be reproduced by Quantum Mechanics, a trajectory or an ensemble of them. Their response to this question was that in a coarse grain approach, the quantum state may behave classically if we consider an ensemble of trajectories. Those results were later confirmed by others [23,24,12,22,21,25,19]. Ballentine and collaborators also argue that the decoherence is not necessary if we take into account the experimental limitations. This is the *Coarse Grained Approach to Classical Limit of Quantum Mechanics*. In fact, both approaches are necessary, since there is a combination of factors that must be considered in order to reproduce the classical regime [25]: large actions, the interaction with the environment and experimental observation limitations. In fact, if Quantum Mechanics domain includes Classical Mechanics domain, then Quantum Mechanics must reproduce all classical experiments and observations, including individual systems like a planet or a star. The action of the measurement apparatus on the system is closely related to the decoherence program [31,32,13–16], but there is a subtle difference: if we consider a situation where the action of the environment is negligible, the system is almost isolated, and if we perform continuous simultaneous measurements of position and momentum, then the information about the quantum nature of the particle will be lost and

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the Newtonian regime is achieved [33,34]. Those results can be summarized in a simple way: decoherence and experimental limitations are responsible for achieving the Liouville classical regime, while the continuous monitoring of the system leads to Newtonian regime [33].

Despite the great advances on the Classical Limit problem, quantifying the degree of classicality of a quantum state is still an open question. In the core of the Decoherence program is the assumption that the environment is usually composed of a large number of particles, thus, due to the thermodynamic limit, the environment (thermal bath) is essentially classical [35,36]. Paradoxically, it has been shown that an interaction with one degree of freedom system can lead the system to behave as it was classical, an example of a small quantum system whose classical counterpart is chaotic and able to produces decoherence-like behavior [37]; similar results can be found in references [38–41]. Oliveira and Magalhães [21] have shown that a single degree of freedom system is, in the context of decoherence, equivalent to a n-degree of freedom system. This equivalence is quantified by the effective Hilbert space size, which is “as the Hilbert-space size of the phase state that generates purity loss equivalently as the other particular environmental states”. Therefore, the effective Hilbert space size is a quantifier of the effectiveness of a system as an environment, i.e. the effectiveness of a specific model mimicking a bath is closely related to the classicality of such state.

Given the richness of possible physical systems and the complicated structure of the quantum state space, it is no surprise that various notions of classicality have been defined. It seems impossible to grasp the variety of quantum states with a unique parameter, especially in infinite-dimensional Hilbert spaces and, therefore, different classicality quantifiers should be considered as complementary rather than competitive.

In the context of harmonic oscillator potential, many classicality quantifiers were defined in terms of how a given state *differs* from a coherent one. These approaches follow from the postulate that coherent states are the only pure classical states in this situation [42–44]. Some examples are Mandel Q-parameter [45] and its various generalizations [46–49]. Another approach is to use the distance of the state to the closest classical state defined in Ref. [50], also used in Refs. [51–56]. These approaches to quantify nonclassicality strongly depend on the chosen set of states used as reference classical states and the norms or metrics used to define the distances. Another quantifier of nonclassicality is based on the convolution of the P-function with the amount of thermal noise needed to get a non-negative phase-space function [57]. Other measures are based on the entanglement potential of non-classical states [58–60]. In Refs. [61,62], the amount of nonclassicality is quantified in terms of the minimal number of coherent states that are needed to be superposed in order to represent the state under study. It is a member of a general class of algebraic measures, applying to different notions of nonclassicality [63]. A moment-based approach was introduced to formulate measurable witnesses for the degree of nonclassicality [64]. Another approach is to determine the degree of nonclassicality based on the Fourier transform of the Glauber–Sudarshan P-function, the characteristic function [65,66]. In reference [67], the authors quantify the classicality of mixed states from the perspective of representation theory of semi-simple Lie groups and give a group theoretic characterization of cases when it is possible to give an explicit, closed form criterion for a mixed state to be classical. Again, the definition of classicality is heavily dependent on the criteria that coherent states are the most classical states.

This approach can not be easily generalized to other potentials, since the coherent states of the harmonic oscillator are not attainable for them and, therefore, cannot serve as the reference set of classical states. The standard coherent states can be generalized for arbitrary potentials in different nonequivalent ways [68,69] and it is not clear which class of states should be considered classical, and hence it is not clear what is the best set of reference states for the determination of the nonclassicality of states in other potentials.

The nonclassicality of quantum states phase space is also connected with measures based on information theory [70–72]. In reference [70], Ferraro et al. show that there are distinct notions of classicality, and, under their considerations, that there exist quantum correlations that are not accessible by information-theoretic arguments. Shahandeh et al. [71] show that the only known classicality criterion violated by a non-local boson sampling protocol [73] is the phase-space nonclassicality. Baumgratz et al. [74] investigated the quantifies of resource theory of quantum coherence. In reference [72] the authors investigated non-classical light, and they show that quantum resource [74] is the same of Glauber [75]; the non-classical light can be interpreted as a form of coherence, their procedure is based on the negativity of P-distribution.

The rest of paper is organized as follows: in section 2 we define a new measure, the Roughness R, and prove that it is bounded between $[0, 1]$. Given two states, we say that the one with a larger value of R is more non-classical than the other. In section 3 we address some important quantum states, and evaluate the Roughness for each one of them. We stress that we could find, both for lower and upper bounds, examples of states that, in limit case, achieve those values. In section 4 we compare the Roughness with another classicality measure, the Negativity N. First, N is not a bounded function, so it can be more difficult to compare any two given states. Also, we show that there are some states with $N = 0$ (said to be totally classical), but with $R > 0$, i.e. the Roughness can find some quantumness in such cases. Particularly, we study a convex mixing between a thermal and a Fock state, and supported by entropy, we show that the Roughness is more reliable, especially for small temperatures. At last, in section 5 we define another classicality measure, the Dynamic Distance Measure D. While R evaluates the inherent quantumness of a state, D quantifies how much a quantum dynamics is far from a classical one. We numerically evaluate both R and D for the quartic model, and we find a complementary behavior between them for such model.

2. Roughness: definition and bounds

The Wigner quasipropability distribution, better known as Wigner function, was introduced in 1932 by Eugene Wigner [76]. It is a real-valued function for any arbitrary quantum state Ψ, and it is given by

$$W_\Psi(q,p) = \frac{1}{2\pi}\int_{\mathbb{R}} dx\, e^{ipx} \left\langle q - \frac{x}{2}\middle|\Psi\right\rangle\left\langle\Psi\middle|q+\frac{x}{2}\right\rangle. \tag{1}$$

As a distribution, it is normalized, i.e.

$$\int_{\mathbb{R}^2} dq\, dp\, W_\Psi(q,p) = 1\,,$$

it is also common to say that it has unitary volume. The Wigner function is a real bounded function, with $|W_\Psi(q,p)| \le \pi^{-1}$ for any $(q,p) \in \mathbb{R}^2$. Moreover, it is square integrable

$$\|W_\Psi\|^2 = (W_\Psi, W_\Psi) = \int_{\mathbb{R}^2} dq\, dp\, [W_\Psi(q,p)]^2 \le \frac{1}{2\pi}\,, \tag{2}$$

and the equality above holds when Ψ is a pure state [77]. The inner product above is the canonical one in the $\mathcal{L}^2(\mathbb{R}^2)$ space. Among its properties, we emphasize the fact that $W_\Psi(q,p)$ can