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# Dimension improvement in Dhar's refutation of the Eden conjecture

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ABSTRACT

We consider the Eden model on the $d$-dimensional hypercubical unoriented lattice, for large $d$. Initially, every lattice point is healthy, except the origin which is infected. Then, each infected lattice point contaminates any of its neighbours with rate 1. The Eden model is equivalent to first passage percolation, with exponential passage times on edges. The Eden conjecture states that the limit shape of the Eden model is a Euclidean ball. By pushing the computations of Dhar [5] a little further with modern computers and efficient implementation we obtain improved bounds for the speed of infection. This shows that the Eden conjecture does not hold in dimension superior to 22 (the lowest known dimension was 35).



## 1. The Eden model: definitions and previous results

We consider the first passage percolation on a $d$-dimensional hypercubical unoriented lattice ([2]) as stated in [3]. Let $\{\alpha(x,y)|(x,y)\in \text{edges of } \mathbb{Z}^d\}$ be a family of i.i.d random variables, with exponential law of parameter 1. Let $n\in\mathbb{N}$. For a path $\mathcal{W}$: $x_0 \to x_1 \to \ldots \to x_n$ of neighbouring vertices, we define the passage time along $\mathcal{W}$: $\alpha(\mathcal{W})=\sum_{i=1}^{n}\alpha(x_{i-1},x_i)$. The family $\{\alpha(x,y)|(x,y)\in$ edges of $\mathbb{Z}^d\}$ defines a random distance, $\forall(x,y)\in\mathbb{Z}^d\times\mathbb{Z}^d$

$$\mathcal{D}(x,y)=\inf_{\mathcal{W}\text{ path from } x \text{ to } y}\alpha(\mathcal{W}).$$

For all $t\in\mathbb{R}$ we set $B_t=\{x\in\mathbb{Z}^d|\mathcal{D}(0,x)\le t\}$. Richardson (1973) and Cox–Durett (1981) have shown that there exists a compact convex $B^*\subset\mathbb{R}^d$ such that for all $\epsilon>0$

$$\mathbb{P}\left((1-\epsilon)B^*\subset\frac{B_t^d}{t}\subset(1+\epsilon)B^*,\ \text{for t big enough}\right)=1.$$

Eden conjectured that this limit form $B^*$ was a Euclidean ball in every dimension.

For all $n\in\mathbb{N}$ we note $\mathcal{P}_n^d$ the hyperplane of equation $x_1=n$ in dimension $d$. We note $\mathbf{0}$ the origin of the hypercube. Observe that $\mathcal{D}(\mathbf{0},\mathcal{P}_n^d)$ is the distance between the origin and the hyperplan $\mathcal{P}_n^d$. Cox–Durrett, Hammersley and Welsh ([6]) have shown that with probability 1 $\lim_{n\to+\infty}\frac{\mathcal{D}(\mathbf{0},\mathcal{P}_n^d)}{n}=\mu_{axis}^d$, for a certain $\mu_{axis}^d$, and moreover that

$$\mu_{axis}^d=\inf_{n\to+\infty}\frac{\mathbb{E}(\mathcal{D}(\mathbf{0},\mathcal{P}_n^d))}{n}.$$

Dhar ([5]) obtained numerical upper bounds on $\mathbb{E}[\mathcal{D}(\mathbf{0},\mathcal{P}_n^d)]$ for small $n=1,2$ and valid for any $d$. This yields good upper bounds for $\mu_{\text{axis}}^d$. Due to computer limitations, Dhar was not able to use his method in 1988 for $n>2$. The aim of this letter is to detail how to extend his computations, and how to compute efficiently new upper bounds for $\mu_{\text{axis}}^d$.

For all $n\in\mathbb{N}$ we note $\mathcal{J}_n^d$ the hyperplane $\mathcal{J}_n^d=\{x_1+x_2+\ldots+x_d=\lfloor n\sqrt{d}\rfloor\}$. Observe that $\mathcal{J}_n^d$ is chosen so that it is at the same Euclidean distance from $\mathbf{0}$ as $\mathcal{P}_n^d$. Therefore, if the Eden conjecture were true, one would have $\mathcal{D}(\mathbf{0},\mathcal{P}_n^d)=\mathcal{D}(\mathbf{0},J_n^d)+o(n)$. For the same reasons as before $\lim_{n\to+\infty}\frac{\mathcal{D}(\mathbf{0},J_n^d)}{n}$ exists and thus we can define $\mu_{diag}^d$ as $\mu_{diag}^d=\lim_{n\to+\infty}\frac{\mathcal{D}(\mathbf{0},J_n^d)}{n}$. Couronné, Enriquez and Gerin ([3]) found an numerical lower bound on $\mu_{diag}^d$ (in fact, the same result also appears in a different form in [4]):

$$\mu_{diag}^d\ge\frac{0.3313\ldots}{\sqrt{d}}.$$

This means that a lower bound on the time of infection along the diagonal has been found. By combining his results with Dhar, [3] showed that $\mu_{axis}^{35}<\mu_{diag}^{35}$, and thus that the limit shape of the infection $B^*$ is not an euclidean ball in dimension 35. In this letter we extend Dhar's method and use [3] lower bound to prove that $\mu_{axis}^{22}<\mu_{diag}^{22}$.

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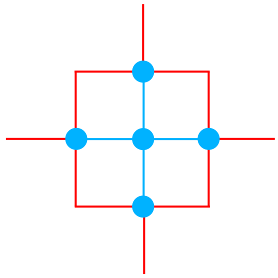

**Fig. 1.** The cluster $C$ (in blue) and the set of perimeter bounds $S$ (in red). (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

**Theorem 1.** *$\mu^{22}_{axis} < \mu^{22}_{diag}$. In particular the limiting shape $B^*$ is not an Euclidean ball, and the Eden conjecture is false in dimension 22.*

## 2. Dhar's strategy for $\mathcal{P}^d_1$

**Idea** Since we will push Dhar's strategy a little further, we first detail the idea introduced in [5]. To compute an upper bound for $\mathbb{E}([\mathcal{D}(\mathbf{0}, \mathcal{P}^d_n)])$, Dhar slightly modifies the model and considers a *unidirectional* infection. This means that a site in $\mathcal{P}^d_i$ can only contaminate its neighbours in $\mathcal{P}^d_i$ and $\mathcal{P}^d_{i+1}$. We note $\tau^d_n$ the time of infection from $\mathbf{0}$ to the plan $\mathcal{P}^d_n$ by the *unidirectional* infection. It is clear that this infection spreads more slowly than the original model, and therefore we obtain for every $n$

$$\mu^d_{\text{axis}} \leq \frac{\mathbb{E}([\mathcal{D}(\mathbf{0}, \mathcal{P}^d_n)])}{n} \leq \frac{E[\tau^d_n]}{n}.$$

From now on, we only consider the model of unidirectional infection in our computations.

### 2.1. Notations

We consider a $d$-dimensional infection. Let $T(C) = \mathbb{E}(\tau^d_1 | B_0 = C)$ be the expected waiting time before the infection reaches $\mathcal{P}^d_1$ starting from an infected cluster $C$ in $\mathcal{P}^d_0$ (and the other sites are healthy). For a cluster $C \subset \mathcal{P}^d_0$ of $i$ sites, we define $S$ its set of perimeter edges (see Fig. 1). As stated in [5] we have

$$|S| \geq \lceil 2(d-1) i^{\frac{d-2}{d-1}} \rceil := s_i. \tag{1}$$

For an edge $e = (x, y)$ such that $x \in C$ and $y \notin C$, we set $v^+(e) = y$ (*i.e.* $v^+(e)$ is the endpoint of $e$ which is not in $C$). We define $T_i = \max_{|C|=i} T(C)$.

### 2.2. Recursive inequality

Let $t_1$ be the time at which the first contamination occurs. At time $t_1$, a site in cluster $C$ contaminates either one of its neighbours in $\mathcal{P}^d_0$ or one of its neighbours in $\mathcal{P}^d_1$ (see Fig. 2). The total number of such “susceptible” sites is given by $|S| + i$, it follows that $t_1 \stackrel{\text{def}}{=} \min\{z_1, \dots, z_{|S|+i}\}$ where $z_i$'s are i.i.d. passage times, *i.e.* $t_1$ is distributed as an exponential r.v. with mean $1/(|S| + i)$. Using the same argument we have:

$$T_i \leq \frac{1}{i}. \tag{2}$$

At time $t^+_1$, the new infected site $x$ is uniformly distributed among the $|S| + i$ possibilities. If $x \in \mathcal{P}^d_1$ then $\tau_1 = t_1$. If $x \in \mathcal{P}^d_0$, then the infection goes on, starting from configuration $C' = C \cup \{x\}$. Because of the memoryless property of the exponential distribution, we have the Markov property

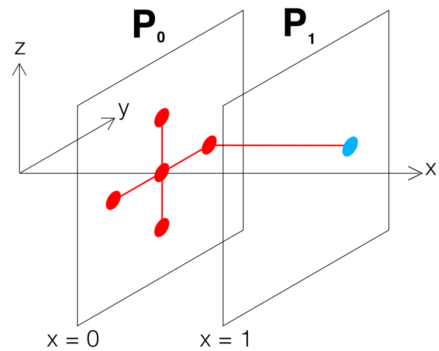


**Fig. 2.** Example of infection. In red the starting cluster $C$. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

$$T(C) - t_1 \mid \{x \text{ is infected at time } t_1\} \stackrel{\text{def}}{=} T(C \cup \{x\}).$$

Therefore we obtain

$$\begin{aligned}
T(C) &= \underbrace{\mathbb{E}(t_1 | B_0 = C)}_{\frac{1}{|S|+i}\,:\,\text{min of } |S|+i \text{ exponential r.v}} + \mathbb{E}\left(\tau^d_1 - t_1 \middle| B_0 = C\right) \\
&= \frac{1}{|S|+i} + \sum_{e \text{ leaving } C} \mathbb{E}\left(\tau^d_1 - t_1 \middle| B_0 = C, B_{t_1} = C'\right) \\
&\quad \times \mathbb{P}\left(C' = C \cup v^+\{e\} \middle| B_0 = C\right) \\
&= \frac{1}{|S|+i} + \sum_{\substack{e \text{ leaving } C \\ v^+\{e\} \in \mathcal{P}_0}} \mathbb{E}\left(\tau^d_1 - t_1 \middle| B_0 = C, B_{t_1} = C'\right) \\
&\quad \times \mathbb{P}\left(C' = C \cup v^+\{e\} \middle| B_0 = C\right) \\
&\quad + \sum_{\substack{e \text{ edge leaving } C \\ v^+\{e\} \in \mathcal{P}_1}} \underbrace{\mathbb{E}\left(\tau^d_1 - t_1 \middle| B_0 = C, B_{t_1} = C'\right)}_{0\,:\,\tau^d_1 = t_1 \text{ since we have reached } \mathcal{P}_1} \\
&\quad \times \mathbb{P}\left(C' = C \cup v^+\{e\} \middle| B_0 = C\right) \\
&= \frac{1}{|S|+i} + \sum_{\substack{e \text{ edge leaving } C \\ v^+\{e\} \in \mathcal{P}_0}} \underbrace{\mathbb{E}\left(\tau^d_1 - t_1 \middle| B_0 = C, B_{t_1} = C'\right)}_{\mathbb{E}\left(\tau^d_1 \middle| B_0 = C'\right) \text{ by Markov property}} \\
&\quad \times \underbrace{\mathbb{P}\left(C' = C \cup v^+\{e\} \middle| B_0 = C\right)}_{\frac{1}{|S|+i}\,:\,\text{choice of one edge among } |S|+i} \\
&= \frac{1}{|S|+i} \left(1 + \sum_{e \text{ edge leaving } C} T\left(C \cup v^+\{e\}\right)\right).
\end{aligned}$$

We have

$$T(C) \leq \frac{1 + |S|\, T_{i+1}}{|S| + i}. \tag{3}$$

The right-hand side is decreasing in $|S|$ and $|S| \leq s_i$ thus

$$T_i \leq \frac{1 + s_i T_{i+1}}{s_i + i}, \tag{4}$$

which is inequality (8) in [5] (note a small misprint in Dhar's inequality (8)). This recursive inequality and a rough bound on $T_n$ leads to a tight bound on $T_1$:

$$\mathbb{E}(\tau^d_1) = T_1 \leq \frac{1 + s_1 T_2}{1 + s_1} \leq \frac{1 + s_1 \frac{1 + s_2 T_3}{2 + s_2}}{1 + s_1}$$