Physics Letters A ••• (••••) •••–•••

Contents lists available at ScienceDirect

Physics Letters A

www.elsevier.com/locate/pla

Complementarity and nonlocality in two-qudit systems

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ARTICLE INFO

Article history:
Received 19 October 2017
Received in revised form 28 December 2017
Accepted 29 December 2017
Available online xxxx
Communicated by M.G.A. Paris

Keywords:
Complementarity
Nonlocality
CHSH experiment
Bell operator
Local and global observables

ABSTRACT

Complementarity and nonlocality are two characteristic traits of quantum physics that distinguish it from classical physics. In this paper, for the two-qubit case, we see that the complementarity between global and local observables in Bell's experiment sets a decisive foundation for the nonlocality of composite systems. We use the Hilbert–Schmidt norm on the commutator of two observables to quantify complementarity between them. Based on the CHSH experiment, we define a measure of complementarity $\mathcal{M}_d$ for the two-qubit case, and extend it to two-qudit systems. Furthermore, we obtain an upper bound on $\mathcal{M}_d$ that scales linearly in the Hilbert space dimension of the qudit.

1. Introduction

Quantum theory was conceived, in the early twentieth century, to explain physical phenomena observed at atomic and sub-atomic scales which classical physics could not account for. After a century, it has become a full-fledged fundamental theory of nature at the microscopic level. Since its advent, it has continuously surprised us, by showing to us what can never be expected in the classical domain. Among several seminal developments, complementarity [1] and nonlocality [2] are eminent concepts. The notion of complementarity has been used in a variety of ways, denoting different concepts and relationships [3]. The complementarity principle restricts joint measurement of certain physical observables in quantum mechanics. It refers to the situation in which two observables cannot have definite values simultaneously. Bohm refers complementarity to pairs of variables by stating [4]: *at the quantum level, the most general physical properties of any system must be expressed in terms of complementary pairs of variables, each of which can be better defined only at the expense of a corresponding loss in the definition of the other*. The wave-particle duality of a quantum system is a well-known instance. In this paper, by complementarity we mean the non-commutativity of two quantum observables, $[X, Y] = XY - YX \neq 0$. For example, in quantum mechanics, position and momentum of a physical system are complementary observables. It is believed that a pair of complementary observables cannot be observed or measured *precisely* at the same moment. However, there are works where it has been shown that complementary observables can be measured simultaneously [5–7]. On the other hand, nonlocality—an exotic feature of quantum physics—has proved an indispensable resource for quantum information processing tasks, including communication and computation [8–10]. Phenomena like quantum teleportation [11] and superdense coding [12], which are not observed in the classical domain, rely heavily on the nonlocality feature of quantum mechanics. It is well-known that jointly measurable observables cannot lead to a violation of any Bell inequality. And authors in [13] have shown that any pair of incompatible observables can be used to violate a Bell inequality. On the other hand, it was shown in [14] that "degree" of complementarity plays a role in determining the quantum nonlocality. So, it's natural to explore and discuss the connection between complementarity and nonlocality. It should be noted that in the case of binary observables, commutators and anticommutators of observables appear in the square of the Bell operator [2,15]. Recently, in Ref. [15], authors have proved tight tradeoffs between the Bell violation and commutation-based incompatibility. They found commutation-based measures a convenient way of expressing relations among more than two observables, and that this formalism recovers several previous results as extreme cases. Appealing advantages with the commutation-based measures are that they are analytic, easily computable, and have a simple physical interpretation. It is believed that the complementarity between global and local observables, in Bell's experiment, leads to nonlo-

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https://doi.org/10.1016/j.physleta.2017.12.053
0375-9601/

cality of composite systems in quantum theory. The connections between complementarity and nonlocality has been explored by several authors [16–18]. Believing in the idea that nonlocality is a consequence of complementarity, to know how nonlocal a composite system can be, it is necessary to know to what extent this complementarity can be. To this end, we need to define a measure of complementarity between two observables. Recall that for two observables X and Y, we use their commutator $[X, Y]$ to show whether they are complementary or not. In this paper, to describe complementarity quantitatively, we consider a norm on these commutators. Norm of a quantity measures its "length". There are a lot of norms defined on matrices. Among them, the family of L_p norms is widely used. Because of finite dimension, all L_p norms are equivalent. Considering this and for the brevity of calculation, we use the L_2 norm (also called the Hilbert–Schmidt norm). So, in the following, we use the Hilbert–Schmidt norm on the commutator of two observables to quantify how complementary they are.

In this paper, we give a definition of complementarity based on the Hilbert–Schmidt norm, study its properties for arbitrary finite-dimensional two-party quantum systems, and seek possible connection between complementarity and nonlocality.

2. CHSH model

In this section, we briefly recall the Clauser, Horne, Shimony, and Holt (CHSH) version of Bell inequality [2,19–21].

2.1. Bell inequality

Quantum mechanics is a nonlocal theory in the sense that it violates Bell inequality—a mathematical inequality involving certain averages of correlations of measurements, derived using the assumptions of locality and realism. That is, quantum mechanics cannot be both local and realistic. Both "locality" and "realism" stem from classical physics. The assumption of locality means that the outcomes of an experiment on a system are independent of the actions performed on a different system which has no causal connection with the first. On the other hand, Einstein's locality states that, even in the case of causal connection, causal influences cannot propagate faster than the speed of light. And reality or determinism means that experiments performed on a system uncover properties that are pre-existing. That is, in an experiment the value of any observable is pre-determined. The experimental setting of Bell's test is as follows. There are two observers, Alice (A) and Bob (B). Each of them has two measurement settings: A_k and B_k, $(k = 1, 2)$. All these observables are dichotomic, i.e., they take values ± 1. The measurement outcomes of these observables are governed by a joint probability distribution. The CHSH version of Bell inequality is expressed as

$$|\langle A_1 B_1 + A_1 B_2 + A_2 B_1 - A_2 B_2\rangle| \leq 2, \tag{2.1}$$

where $\langle XY\rangle = \sum_{i,j} x_i y_j p(x_i, y_j)$. This inequality is valid in any physical theory that is local and realistic, and where the physical observables assume the values ± 1. Now, let A_k and B_k denote the single-qubit Hermitian operators

$$A_k = a^{(k)}.\sigma = \sum_{i=1}^{3} a_i^{(k)} \sigma_i$$

$$B_k = b^{(k)}.\sigma = \sum_{i=1}^{3} b_i^{(k)} \sigma_i,$$

where $a^{(k)}$, $b^{(k)}$ are unit vectors in $\mathbb{R}^3$, and σ_i are Pauli matrices. Recall that if Alice and Bob share the singlet, $|\psi^-\rangle_{AB} = \frac{|01\rangle - |10\rangle}{\sqrt{2}}$, then quantum mechanics says that $\langle A_k B_{k'}\rangle = -a^{(k)}.b^{(k')}$. For the choice of real unit vectors $a^{(1)} = (1, 0, 0)$, $a^{(2)} = (0, 0, 1)$, $b^{(1)} = \frac{1}{\sqrt{2}}(1, 0, 1)$, and $b^{(2)} = \frac{1}{\sqrt{2}}(1, 0, -1)$, quantum mechanics clearly violates the CHSH inequality. The operator $\mathcal{B} = A_1 \otimes (B_1 + B_2) + A_2 \otimes (B_1 - B_2)$ in CHSH inequality (2.1) is called the Bell operator. Note that the Bell operator is a global (or, nonlocal) observable since it is an observable of the whole system. On the other hand, we can define a local observable as $A(r) \otimes B(s) = (\sum_{i=1}^{3} r_i \sigma_i) \otimes (\sum_{j=1}^{3} s_j \sigma_j)$, where r and s are unit vectors in $\mathbb{R}^3$. The Bell operator, in general, does not commute with the local operator. Then, it is natural to use a norm on the commutator of these observables to quantify how complementary they are. In this paper, we use the Hilbert–Schmidt norm of operators to quantify it. The Hilbert–Schmidt norm is unitary invariant with respect to its argument. For the Bell operator $\mathcal{B}$, the quantity

$$\mathcal{M}_{\mathcal{B}} = \sup_{r,s} \| [\mathcal{B},\ A(r) \otimes B(s)] \|_2, \tag{2.2}$$

where r and s run over the unit spherical face of $\mathbb{R}^3$, quantifies the maximal "amplitude" of complementarity of the Bell operator with the local observable.

2.2. Generalized Bell operators for two-qubit case

In the typical CHSH setting, Alice and Bob separately measure a spin along some direction each time. And for a single qubit, nontrivial observables are only spins. Considering this, we define the following operator for two qubit system

$$\mathcal{A} = \sum_{i,j=1}^{3} \alpha_{ij} \sigma_i \otimes \sigma_j, \tag{2.3}$$

where $\alpha_{ij} \in \mathbb{R}$ and σ_i is the canonical Pauli matrix. Since this is an extension of the Bell operator, we call it the generalized Bell operator (this can be viewed as a correlation tensor!). With this development, we are interested in computing the following quantity

$$\mathcal{M} = \sup_{r,s,\alpha_{ij}} \| [\sum_{i,j=1}^{3} \alpha_{ij} \sigma_i \otimes \sigma_j,\ A(r) \otimes B(s)] \|_2, \tag{2.4}$$

where $\alpha_{ij} \in \mathbb{R}$ and unit vectors r, s run over all possible choices. However, we demand $\mathcal{M}$ to be bounded. For this, we normalize α_{ij}: $\sum_{i,j=1}^{3} \alpha_{ij}^2 = 1$. Note that due to the Bloch ball structure, for any single-qubit spin operator $A(r) = \sum_{i=1}^{3} r_i \sigma_i$, we can always find a unitary operator U such that $A(r) = U\sigma_3 U$. Then for any operator $\mathcal{A}$ on the two-qubit system, and single-qubit spin operators $A(r)$ and $B(s)$, we can find single-qubit unitary operators U and V such that

$$\begin{aligned}
&\| [\mathcal{A}, A(r) \otimes B(s)] \|_2 \\
&= \| [\mathcal{A}, U \otimes V(\sigma_3 \otimes \sigma_3) U^\dagger \otimes V^\dagger] \|_2 \\
&= \| U^\dagger \otimes V^\dagger [\mathcal{A}, U \otimes V(\sigma_3 \otimes \sigma_3) U^\dagger \otimes V^\dagger] U \otimes V \|_2 \\
&= \| [U^\dagger \otimes V^\dagger \mathcal{A} U \otimes V, \sigma_3 \otimes \sigma_3] \|_2 \ .
\end{aligned} \tag{2.5}$$

Let $\mathcal{A}$ be the generalized Bell operator (2.3). Since Pauli matrices $\{\sigma_i\}_{i=1}^{3}$ are traceless, hermitian and orthogonal with $tr(\sigma_i \sigma_j) = 2\delta_{ij}$, therefore $\{U\sigma_i U^\dagger\}_{i=1}^{3}$ are also traceless, hermitian and orthogonal. Furthermore, when

Please cite this article in press as: H. Wang et al., Complementarity and nonlocality in two-qudit systems, Phys. Lett. A (2018), https://doi.org/10.1016/j.physleta.2017.12.053