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Adiabatic dynamics of one-dimensional classical Hamiltonian dissipative systems

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ABSTRACT

A linearized plane pendulum with the slowly varying mass and length of string and the suspension point moving at a slowly varying speed is presented as an example of a simple 1D mechanical system described by the generalized harmonic oscillator equation, which is a basic model in discussion of the adiabatic dynamics and geometric phase. The expression for the pendulum geometric phase is obtained by three different methods. The pendulum is shown to be canonically equivalent to the damped harmonic oscillator. This supports the mathematical conclusion, not widely accepted in physical community, of no difference between the dissipative and Hamiltonian 1D systems.

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1. Introduction

Dynamics of our world is governed and described by differential equations. Realization of this startling fact was evaluated by Newton as the most important discovery of his life. However, explicit analytical solutions of differential equations are the exception rather than the rule. This makes scientists develop special and approximate methods for the analysis of differential equations because every new step in understanding the properties of their solutions gives a further insight into a physical theory described by corresponding equations. Thus, for example, the discovery of adiabatic invariants of the second order differential equation with slowly varying parameters was an important step in the development of quantum theory. The existence of one more remarkable property of this equation, the so-called geometric phase, was noticed only 80 years later. Historical aspects of the development of ideas related to the understanding of the properties of solutions of differential equations with slowly varying parameters as well as their theoretical, experimental and applied aspects one can find in many reviews and books (see, for example, [1–5]).

The quantity considered in the present paper, the geometric phase, is also known as the topological or nonholonomic phase and often associated with the names of its pioneers: Rytov, Vladimirskii, Pancharatnam, Berry, Hannay, less frequently with Ishlinskii (who gave the explanation of systematic gyroscope bias error after a long voyage), and others. In our work we consider this concept at the classical, non-quantum level and in what follows call it the geometric or Hannay phase. The geometric phase can occur both in quantum [6] and in classical [7,8] systems. This is not astonishing in view of the actually identical second order differential equations which are the time-independent Schrödinger equation and the Newton (or Hamilton) equation for the harmonic oscillator with a linear restoring force. The analogy between quantum and classical phenomena is clearly seen, for example, when one compares the classical phenomenon of parametric resonance and the band character of the spectrum of quantum particle in a stationary periodic field: both of the phenomena are described by the Hill equation. This analogy was also repeatedly used in the study and comparison of the adiabatic dynamics of classical systems and the WKB approximation of quantum mechanics [9]. In mathematical terms, the geometric phase is a correction to the dynamical phase for the harmonic solution of a linear differential equation with the broken time-reversal invariance or, in other words, for the solution which describes the vibrational mode of motion of dynamical systems [10,11] with slowly varying parameters.

In the present work we give an elementary example of mechanical system illustrating the physical meaning of Hamiltonian (1) and, in this way, the possible range of applicability of Hannay's [7] results. This mechanical system is a plane mathematical pendulum with the slowly varying mass and string length, and with the sus-

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pension point moving at a slowly varying speed. In Section 3 we derive the expression for geometric phase arising in this system. The fact of canonical equivalence between the considered pendulum and a damped harmonic oscillator we show in Section 4. This fact is somewhat surprising from the physical point of view and trivial, at the same time, from the mathematical point. We discuss this duality at Section 5. A complex form of the GHO Hamilton function is presented in Appendix A. For the sake of completeness and readability we often appeal to already known results with appropriate citation.

2. Generalized harmonic oscillator

The simplest second order equation which can be a demonstrative example of the existence of geometric phase in classical adiabatic dynamics [7,8] is the equation of motion of the generalized harmonic oscillator (GHO). The importance of this example is confirmed by the fact that scientists after Hannay [7] often returned to this equation [8,11–15] using different methods for its analysis. The Hamiltonian of the GHO is given by

$$H = \frac{1}{2}\left(\alpha Q^2 + 2\beta Q P + \gamma P^2\right), \tag{1}$$

where Q and P are the canonically conjugate coordinate and momentum; α, β, and γ are the parameters of the generalized oscillator. When the parameters α, β and γ are constant, the energy of the system is a constant of motion. For the values of α, β and γ satisfying the inequality $\alpha\gamma > \beta^2$, solutions of the Hamilton equations take the form:

$$Q = r\cos\Theta,\ P = -\frac{r}{\gamma}\left(\beta\cos\Theta + \omega\sin\Theta\right),$$
$$\Theta = \omega t,\ \omega = \sqrt{\alpha\gamma - \beta^2}. \tag{2}$$

If the parameters change slowly, see Eq. (7), the motion of the oscillator can be approximately regarded as the periodic one of the form (2) with the slowly varying amplitude r and phase Θ both of which should be determined. In this case, the energy of the system is not conserved, but there is a new approximate conserved quantity, the adiabatic invariant,

$$I = \frac{\omega r^2}{2\gamma} = \frac{E}{\omega}, \tag{3}$$

which remains constant with (non-analytic) exponential accuracy $\Delta I \sim \exp(-1/t_0\epsilon)$ [16], where ϵ^{-1} is the characteristic time scale of slowly changing parameters and t_0 is some constant determined by analytical properties of varying parameters. The phase of the oscillator is an 'almost linear' function of time. It was shown in the works by Hannay [7] and Berry [8] that the phase of the oscillator can be represented as the sum of two quantities, $\Theta = \Theta_d + \Theta_g$, where the dynamic Θ_d and the geometric Θ_g phases are:

$$\Theta_d = \int^t \omega dt,$$
$$\Theta_g = \frac{1}{2}\int^t \frac{\beta}{\omega}\left(\frac{\dot\gamma}{\gamma} - \frac{\dot\beta}{\beta}\right)dt = \frac{1}{2}\int_\Gamma \frac{\beta}{\omega}\left(\frac{d\gamma}{\gamma} - \frac{d\beta}{\beta}\right). \tag{4}$$

The independence of the geometric phase of time follows from the last equation of (4), provided the adiabatic condition holds, and explains the name of this phase. The dependence of the phase on the path Γ of integration is associated with the concept of anholonomy. Reversing the direction of integration along the contour Γ changes the sign of the geometric phase, and when the parameter β, which violates the time-reversal invariance of the Hamiltonian (1), becomes zero, the geometric phase vanishes as well. If the line of variation of the parameters α, β and γ is the closed curve Γ, then the integral corresponding to the geometric phase can be converted by Stokes' theorem to the surface integral which is also independent of the time of parameter change:

$$\Theta_g = \iint_{S(\Gamma)} \frac{1}{4\omega^3}\left(\gamma dS_{\alpha\beta} + \alpha dS_{\beta\gamma} + \beta dS_{\alpha\gamma}\right), \tag{5}$$

where $dS_{\alpha\beta} = d\alpha \wedge d\beta$ and similar expressions are projections of oriented surface elements on relevant directions. This is the essence of Berry and Hannay's results in the application to the classical mechanics. The result (4) can be obtained [12] by averaging over the 'fast' variable Θ without appealing to the action-angle variables, as it was originally made in Hannay's work [7]. In more details this procedure is as follows: it is necessary to substitute the expressions (2) into the Hamiltonian equations of motion,

$$\dot P = -\frac{\partial H}{\partial Q}, \quad \dot Q = \frac{\partial H}{\partial P}, \tag{6}$$

bearing in mind that the parameters α, β and γ are functions of time. Solving the obtained equations with respect to $\dot r$ and $\dot\Theta$ and averaging them over the period of motion, one arrives at simple differential equations for α, β and γ that have the solutions given by the expressions (3) and (4). Another alternative method of obtaining the results (3) and (4) one can find in Appendix B.

Note, that the geometric phase Θ_g stems from the non-potential (vortex) nature of the differential form $\frac{\beta}{\omega}\left(\frac{d\gamma}{\gamma} - \frac{d\beta}{\beta}\right)$, since in general case $\frac{\partial}{\partial\gamma}\left(-\frac{1}{\omega}\right) \neq \frac{\partial}{\partial\beta}\left(\frac{\beta}{\gamma\omega}\right)$; in the theory of differential forms such forms are called inexact. The Hannay phase cannot be calculated only on the basis of initial and final states of the oscillator and depends on the path connecting the start $(\alpha_s, \beta_s, \gamma_s)$ and end $(\alpha_e, \beta_e, \gamma_e)$ points-states of the system in the parameter space (α, β, γ). For the existence of the geometric phase (see Eq. (4)), the most significant factor is the lack of T-invariance of Hamiltonian (1).

In spite of the simplicity, this result had a great influence on the subsequent development of the theory of dynamical systems and found numerous applications [1–5]. However, until now the question, which systems can be described by the Hamiltonian of the GHO is still open. In the work [12] it was shown that the Hamiltonian (1) is canonically equivalent to the Hamiltonian of the equation of damped harmonic oscillator. The result, on the one hand, is a bit surprising but, on the other hand, leaves a feeling of dissatisfaction. In particular, the existence of other simple counterparts of the GHO among well-known mechanical systems seems to be natural.

3. Plane mathematical pendulum

Let us consider the motion of simple plane linearized mathematical pendulum with the suspension point moving with a small acceleration along the vertical axis OY of the oscillation plane XOY, see Fig. 1. The speed v of the suspension point as well as two other pendulum parameters – the mass m and the length l of the string – are supposed to be slowly changing functions of time with the characteristic scale ϵ^{-1} much greater than the period T of harmonic oscillations of the pendulum:

$$\epsilon T << 1. \tag{7}$$

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