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On a reduction of the generalized Darboux–Halphen system

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ABSTRACT

The equations for the general Darboux–Halphen system obtained as a reduction of the self-dual Yang–Mills can be transformed to a third-order system which resembles the classical Darboux–Halphen system with a common additive terms. It is shown that the transformed system can be further reduced to a constrained non-autonomous, non-homogeneous dynamical system. This dynamical system becomes homogeneous for the classical Darboux–Halphen case, and was studied in the context of self-dual Einstein's equations for Bianchi IX metrics. A Lax pair and Hamiltonian for this reduced system is derived and the solutions for the system are prescribed in terms of hypergeometric functions.

1. Introduction

The Darboux–Halphen differential equations often referred to as the classical Darboux–Halphen (DH) system

$$\dot{\omega}_i = \omega_j\omega_k - \omega_i\left(\omega_j + \omega_k\right), \qquad i \neq j \neq k = 1, 2, 3, \ \textit{cyclic}, \qquad \dot{} := \frac{d}{dt}, \tag{1.1}$$

was originally formulated by Darboux [1] and subsequently solved by Halphen [2]. The general solution to equation (1.1) may be expressed in terms of the elliptic modular function. In fact Halphen related the DH equation in terms of the null theta functions.

The system (1.1) has found applications in mathematical physics in relation to magnetic monopole dynamics [3], self dual Einstein equations [4,5], topological field theory [6] and reduction of self-dual Yang–Mills (SDYM) equations [7]. Recently in [8], the DH system was reviewed from the perspective of the self-dual Bianchi-IX metric and the SDYM field equations, describing a gravitational instanton in the former case, and a Yang–Mills instanton in the latter. All systems related to the DH system such as Ramanujan and Ramamani system were covered, as well as aspects of integrability of the DH system.

Ablowitz et al. [9,10] studied the reduction of the SDYM equation with an infinite-dimensional Lie algebra to a 3×3 matrix differential equation. This work led to a generalized Darboux–Halphen (gDH) system which differs from the DH system by a common additive term. The gDH system was also solved originally by Halphen [11] in terms of general hypergeometric functions and whose general solution admits movable natural barriers which can be densely branched.

In this article, we discuss certain aspects related to the integrability of the gDH system. Some of these features were implicit in the original formulation of the system but were never made concrete. Specifically, we show that it is possible to derive naturally from the gDH system yet another reduced system of equations which satisfy a constraint. This constrained system resembles a non-autonomous Euler equation similar to that derived by Dubrovin [12] but with non-homogeneous terms. Furthermore, we derive a simple Lax pair for the constrained system. The paper is organized as follows. In Section 2, the gDH system is introduced and a constrained system is derived from it. Then the solutions of both the gDH and the constrained systems are discussed. In Section 3, we derive following [10], the gDH system from a ninth-order dynamical system that is obtained as a reduction of the SDYM field equations equation. We provide some details in our derivation that were not included in earlier papers. Then we discuss the constrained system in the framework of a fifth-order system that arise as a special case of the SDYM reduction. In Section 4, we formulate a Lax pair and a Hamiltonian for the reduced system introduced in Section 2.

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2. The gDH system

In this section, we introduce the gDH system for the complex functions $\omega_i(t)$

$$\dot{\omega}_i = \omega_j \omega_k - \omega_i \left(\omega_j + \omega_k\right) + \tau^2 , \quad i \neq j \neq k = 1, 2, 3, \textit{ cyclic} . \tag{2.1}$$

The common additive term τ^2 is elaborated as

$$\tau^2 = \alpha_1^2 x_2 x_3 + \alpha_2^2 x_3 x_1 + \alpha_3^2 x_1 x_2 \qquad \text{with} \qquad x_i = \omega_j - \omega_k ,$$
$$i \neq j \neq k, \textit{ cyclic}, \qquad x_1 + x_2 + x_3 = 0 , \tag{2.2}$$

where α_i, $i = 1, 2, 3$ are complex constants. As mentioned in Section 1, the gDH system arises from a particular reduction of the SDYM equations [9,10]. They also appear in the study of $SU(2)$-invariant, hypercomplex four-manifolds [13]. In Section 3, we will provide a derivation of the gDH system from the SDYM reductions following [10].

In the following, we derive from (2.1) a reduced system of differential equations which satisfy a constraint.

2.1. Constrained gDH system

Note that the variables x_i defined in (2.2) satisfy the equations

$$\dot{x}_i = -2\omega_i x_i , \quad i = 1, 2, 3 , \tag{2.3}$$

which are obtained from (2.1) by taking the difference of the equations for ω_j and ω_k. Using (2.3), the gDH equations (2.1) can be re-expressed as follows:

$$\dot{\omega}_i - \frac{\omega_i}{2}\left(\frac{\dot{x}_j}{x_j} + \frac{\dot{x}_k}{x_k}\right) = \omega_j \omega_k + \tau^2 .$$

Then by defining new variables W_i, $i = 1, 2, 3$ via

$$W_i := \frac{\omega_i}{\sqrt{x_j x_k}} , \quad i \neq j \neq k, \textit{ cyclic} , \tag{2.4}$$

one obtains the system

$$\dot{W}_i = x_i W_j W_k + \frac{\tau^2}{\sqrt{x_j x_k}} . \tag{2.5}$$

It follows from (2.5) that

$$\sum_{i=1}^{3} W_i \dot{W}_i = W_1 W_2 W_3 \sum_{i=1}^{3} x_i - \frac{\tau^2}{2x_1 x_2 x_3} \sum_{i=1}^{3} \dot{x}_i = 0$$

after using (2.4), (2.3) and the fact that $x_1 + x_2 + x_3 = 0$. Thus, one finds that the quantity

$$Q := \sum_{i=1}^{3} W_i^2 = \frac{\omega_1^2}{x_2 x_3} + \frac{\omega_2^2}{x_1 x_3} + \frac{\omega_3^2}{x_1 x_2}$$

is a constant. However, the quantity Q is *not* a conserved quantity of (2.5), rather $Q = -1$ is an identity which follows from the definition of the variables W_i in (2.4). Indeed, a direct calculation using $x_1 + x_2 + x_3 = 0$, shows that

$$\begin{aligned} Q &= \frac{\omega_1^2 x_1 + \omega_2^2 x_2 + \omega_3^2 x_3}{x_1 x_2 x_3} = \frac{\omega_1^2 x_1 + \omega_2^2 x_2 - \omega_3^2 (x_1 + x_2)}{x_1 x_2 x_3} \\ &= \frac{x_1(\omega_1 - \omega_3)(\omega_1 + \omega_3) + x_2(\omega_2 - \omega_3)(\omega_2 + \omega_3)}{x_1 x_2 x_3} \\ &= \frac{x_1 x_2 (\omega_2 - \omega_1)}{x_1 x_2 x_3} = -\frac{x_1 x_2 x_3}{x_1 x_2 x_3} = -1 \end{aligned}$$

Therefore, the system in (2.5) is a reduction of the original gDH system; the reduced system can be regarded as a third order system for the W_i satisfying the constraint $Q = -1$. Note that the DH equations (1.1) being a special case ($\alpha_i = 0$) of (2.1), also admits the same reduced system (2.5) as above but with $\tau = 0$.

Remark. A third order system similar to (2.5) but without the non-homogeneous term, was introduced in [14,15] where the authors derived a family of self-dual, SU(2)-invariant, Bianchi-IX metrics obtained from solutions of a special Painleve-VI equation. In that case, the vanishing of the anti-self-dual Weyl tensor and scalar curvature led to a sixth order system described by the classical DH system (1.1) coupled to another third order system. The W_i variables represented different quantities in [14,15] although they were defined in the same way as in (2.4). The quantity Q was a first integral (instead of a number) in that case, depending on the initial conditions for the sixth order system. This sixth order system considered in [14,15] also admits a special reduction to the third order DH system when the metric is self-dual Einstein. It is this latter case which corresponds to the homogeneous version of (2.5) above with $Q = -1$.

Next, we discuss the solution of the reduced system via the solutions of the original gDH system (2.1).

2.2. Solutions

As mentioned in Section 1, Halphen [11] solved the gDH system and expressed its solution in terms of the general hypergeometric equation. Below we discuss a method of solution first given by Brioschi [16].

Let us first introduce a function $s(t)$ via the following ratio:

$$s = \frac{\omega_3 - \omega_2}{\omega_1 - \omega_2} = -\frac{x_1}{x_3} . \tag{2.6}$$

Taking the derivative of $\ln s$ in (2.6) and then using (2.3), the x_i can be written as

$$x_1 = -\frac{1}{2}\frac{\dot{s}}{s-1} , \qquad x_2 = \frac{1}{2}\frac{\dot{s}}{s} , \qquad x_3 = \frac{1}{2}\frac{\dot{s}}{s(s-1)} . \tag{2.7}$$

Using (2.3) once more, the gDH variables ω_i can be expressed in terms of $s, \dot{s}$ and $\ddot{s}$ as

$$\omega_1 = -\frac{1}{2}\frac{d}{dt}\left[\ln\left(\frac{\dot{s}}{s-1}\right)\right] , \qquad \omega_2 = -\frac{1}{2}\frac{d}{dt}\left[\ln\left(\frac{\dot{s}}{s}\right)\right] ,$$
$$\omega_3 = -\frac{1}{2}\frac{d}{dt}\left[\ln\left(\frac{\dot{s}}{s(s-1)}\right)\right] . \tag{2.8}$$

Substituting the above expressions for ω_i into the gDH system (2.1) yields the following third order equation for $s(t)$

$$\frac{\dddot{s}}{\dot{s}} - \frac{3}{2}\left(\frac{\ddot{s}}{\dot{s}}\right)^2 + \frac{\dot{s}^2}{2}\left[\frac{1-\alpha_1^2}{s^2} + \frac{1-\alpha_2^2}{(s-1)^2} + \frac{\alpha_1^2 + \alpha_2^2 - \alpha_3^2 - 1}{s(s-1)}\right] , \tag{2.9}$$

also known as the Schwarzian equation. Equation (2.9) can be *linearized* in terms of the hypergeometric equation as follows. Let $\chi_1(s)$ and $\chi_2(s)$ be any two linearly independent solution of the hypergeometric equation

$$\chi'' + \left(\frac{1-\alpha_1}{s} + \frac{1-\alpha_2}{s-1}\right)\chi' + \frac{(\alpha_1 + \alpha_2 - 1)^2 - \alpha_3^2}{4s(s-1)}\chi = 0 . \tag{2.10}$$

If the independent variable t in the gDH system is defined by