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A graph with fractional revival

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ABSTRACT

An example of a graph that admits balanced fractional revival between antipodes is presented. It is obtained by establishing the correspondence between the quantum walk on a hypercube where the opposite vertices across the diagonals of each face are connected and, the coherent transport of single excitations in the extension of the Krawtchouk spin chain with next-to-nearest neighbour interactions.

1. Introduction

This article provides an example of a graph that exhibits balanced fractional revival (FR) at two sites. FR [1] is observed during a quantum walk process when an initially localized state evolves so that after some time it has non-vanishing probability amplitude uniquely at a number of isolated places. FR is said to be balanced if these probabilities are equal at each site. This phenomenon has been shown to occur in spin chains [2,3]. In these systems, a state having at first a single excitation at one end is found, after finite time, with non-zero amplitudes for this spin up, only at both ends. Balanced FR can thus serve as a mechanism to generate maximally entangled states.

A special case of fractional revival is when the revival takes place at a single site (with probability one). If this site is the one where the spin up was initially located, perfect return occurs. Otherwise, if it is a different site (for instance the opposite end of the chain), one speaks of perfect state transfer (PST). PST in spin chains is attracting much interest [4–6] especially for the design of quantum wires that would require a minimum of external control. PST can also be realized in photonic lattices, i.e. in arrays of optical waveguides [7,8]. It has been shown that non-uniform couplings are required in these devices for PST beyond a few sites [9]. Favoured models are those where the couplings between the nearest neighbours are related to the recurrence coefficients of the Krawtchouk polynomials; they are often referred to by that name for that reason [10].

These advances have prompted the study of PST in spin networks; here the one excitation Hamiltonian is provided by the adjacency matrix of the underlying graph. PST on graphs has been much analysed and the reader is invited to read [11] and [12] for reviews. Little consideration has been given to FR on graphs however. A significant observation that has been made [9,13] is that PST between antipodes of the one-link hypercube is tantamount to end-to-end perfect transport in the Krawtchouk spin chain. This is established by showing that the quantum walk on the graph projects onto the one on the line or P_N, with appropriate weights for the N links. PST occurs in the Krawtchouk chain with nearest-neighbour (NN) couplings; it is however easy to check that FR is not possible in this model. In view of the preceding point, the same goes for the hypercube, namely this graph will not support FR.

As it turns out, an extension of the NN-Krawtchouk model that includes interactions between next-to-nearest neighbours (NNN) has been designed recently [14] and found to admit balanced FR under certain conditions. We shall use these results here, to find a graph with FR, by showing that the quantum walk between antipodal points on that graph projects equivalently to the NNN one-excitation dynamics of the chain known to have FR. This graph will be identified as a hypercube where the opposite vertices across the diagonals on each face are connected and where all these links

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have the same weight relative to the one attributed to the edges of the hypercube.

The remainder of the paper will proceed as follow. We shall first remind the reader of the NNN extension of the Krawtchouk spin chain and of its PST and FR properties. We shall follow by reviewing basic features of Hamming graphs and their adjacency matrices. Their relation to the Krawtchouk polynomials will be brought up. We shall further recall how the connection between quantum walks on the hypercube and on the one-dimensional graph with Krawtchouk weights is made by restricting the adjacency matrix to an appropriate "column" subspace. We shall then determine the graph whose restriction to that subspace yields the NNN one-excitation dynamics. Concluding remarks will be offered and the paper will end with an Appendix where the occurrence of FR on the graph is verified directly.

2. Fractional revival in the Krawtchouk spin chain with next-to-nearest neighbour interactions

We shall consider a spin chain with the following Hamiltonian of type XX on $(\mathbb{C})^{\otimes N}$ where N spins interact with their nearest and next-to-nearest neighbours:

$$H = \frac{1}{2}\sum_{n=1}^{N} J_n^{(1)}(\sigma_n^x\sigma_{n+1}^x + \sigma_n^y\sigma_{n+1}^y) + J_n^{(2)}(\sigma_n^x\sigma_{n+2}^x + \sigma_n^y\sigma_{n+2}^y) + B_n\sigma_n^z \tag{1}$$

where $J_N^{(1)} = 0$ and $J_N^{(2)} = J_{N-1}^{(2)} = 0$. As usual, σ_n^x, σ_n^y and σ_n^z stand for the Pauli matrices with the index n indicating on which of the $\mathbb{C}$ factors they act.

The coupling constants are built from the recurrence coefficients of the Krawtchouk polynomials [15]. (See (19) below.) Let

$$J_n = \frac{1}{2}\sqrt{n(N-n)}. \tag{2}$$

The nearest-neighbour (NN) couplings are taken to be

$$J_n^{(1)} = \beta J_n \tag{3}$$

with β an energy scale parameter. The next-to-nearest neighbour (NNN) couplings are chosen as

$$J_n^{(2)} = \alpha J_n J_{n+1} \tag{4}$$

with α another parameter. Finally, the Zeeman terms will be specified by

$$B_n = \alpha(J_n^2 + J_{n-1}^2). \tag{5}$$

Observe that when $\alpha = 0$, $J_n^{(2)} = B_n = 0$ and the NN Krawtchouk model with no magnetic field is recovered. Note also that slight changes have been made with respect to [14]: we are here taking N (instead of $N+1$) to be the total number of sites and have modified the range of n accordingly.

Owing to rotational symmetry around the z-direction, H preserves the total number of spins that are up along the chain. It will suffice here to consider chain states that have only one excitation or spin up. A natural basis for that subspace is given by the vectors

$$|n\rangle = (0, 0, \ldots, 0, 1, \ldots 0)^T \qquad n = 1, \ldots, N \tag{6}$$

with the only 1 in the nth position. This vector is associated to a single spin up at the nth site. The action of H on those state is

$$H|n\rangle = J_n^{(2)}|n+2\rangle + J_n^{(1)}|n+1\rangle + B_n|n\rangle + J_{n-1}^{(1)}|n-1\rangle + J_{n-2}^{(2)}|n-2\rangle. \tag{7}$$

We remark that if we define the matrix J by

$$J|n\rangle = J_n|n+1\rangle + J_{n-1}|n-1\rangle \tag{8}$$

we have

$$H|n\rangle = (\alpha J^2 + \beta J)|n\rangle. \tag{9}$$

Fractional revival (FR) at two sites occurs if there is a time τ_{FR} such that

$$e^{-i\tau_{FR}H}|n\rangle = \mu|0\rangle + \nu|N\rangle \tag{10}$$

with $\mu, \nu \in \mathbb{C}$ such that $|\mu|^2 + |\nu|^2 = 1$. In other words, FR takes place if the dynamics allows to evolve the state with a spin up localized at site 1 into a state that is "revived" at both ends of the chain. FR is balanced when $|\mu| = |\nu| = \frac{1}{\sqrt{2}}$ in which case, a maximally entangled state has been generated at time τ_{FR}. The special case of FR that happens when $\mu = 0$ is referred to as perfect state transfer (PST) since the spin up at site 1 is then transported at site N with probability one after a time we will denote τ_{PST}. The NN Krawtchouk chain ($\alpha = 0$) is well known to admit PST at $\tau_{PST} = \frac{\pi}{\beta}$.

The coherent transport of single excitation along the NNN Krawtchouk spin has been studied in [14]. In summary, the findings are as follows. It is first noted that FR can not be found in the NN situation when $\alpha = 0$. When $\alpha \neq 0$, balanced FR can happen. Apart from an overall phase factor, μ would be real, ν pure imaginary and $|\mu| = |\nu| = \frac{1}{\sqrt{2}}$. There are conditions on α, β and N.

i. *Case* $\beta \neq 0$:
The ratio $\frac{\alpha}{\beta}$ must be a rational number:

$$\frac{\alpha}{\beta} = \frac{p}{q} \tag{11}$$

where p and q are coprime integers. FR will happen if in addition,
- p is odd
- q and N have different parities.

The time at which balanced FR will then occur first is

$$\tau_{FR} = \frac{\pi q}{2\beta}. \tag{12}$$

PST will be found in these circumstances at double the FR time: $\tau_{PST} = 2\tau_{FR}$. PST is also possible at $\tau_{PST} = \frac{\pi q}{\beta}$ if p is even and q odd even though FR can not be realized in this case.

ii. *Case* $\beta = 0$:
FR and PST as well, are possible only when N is odd. The minimal times are for FR,

$$\tau_{FR} = \frac{\pi}{2\alpha} \tag{13}$$

and for PST, $\tau_{PST} = 2\tau_{FR}$.

3. Elements of the binary Hamming scheme

We shall now review properties of certain Hamming graphs that will be used to obtain a lift to a graph with FR, of the single excitation dynamics of the spin chain with NNN couplings that we have described in the last section.

Recall that a graph $G = (V, E)$ consists of a set V of vertices and of a set E of edges that are two-element subsets of V. Edges might be assigned weights. With $|V|$ the cardinality of V, the adjacency matrix of a graph G is the $|V| \times |V|$ matrix whose A_{xy} entry for $x, y \in V$ is equal to the number of edges between the vertices x and y.