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Quantum and classical phenomenological universalities

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A B S T R A C T

The quantum $Q2$ class of phenomenological universalities (PU) is extended to include exact solutions and minimum uncertainty coherent states of Hulthen, Kratzer–Fues, Tipping and generalized Kratzer–Fues–Tipping oscillators. In the dissociation (quasi-quantum) limit the solutions obtained generate the extended $U2$ class of PU including West–Brown–Enquist universal growth curve and temporal (spatial) fractal functions widely used in the field of life sciences and physics. It will be shown that the PU concept seems to be a valuable methodology for deriving the new forms of potential energy functions with possible applications in theoretical analysis of molecular spectra.

1. Introduction

Recently [1] the concept of PU introduced by Castorina, Delsanto, and Guiot (CDG) [2,3] has been extended to include quantum oscillatory phenomena, coherence and supersymmetry. In particular it was proved that the CDG formalism is a hidden form of supersymmetry, which can be employed not only to classify the growth phenomena emerging in the complex systems but also to obtain exact solutions of the quantal time-like Schrödinger and space-like Horodecki–Feinberg [9,8] equations for harmonic and anharmonic oscillators. They can be employed to construct quantum coherent states of the time- and space-dependent harmonic, Morse [4] and Wei [5] oscillators belonging to the $Q0$, $Q1$ and $Q2$ classes of PU. In the dissociation (classical) limit coherent states reduce to the well-know Gompertz [6] and West–Brown–Enquist (WBE)-type (e.g. logistic, exponential, Richards, von Bertalanffy) [7] time-dependent functions of growth or space-dependent distribution functions characterized as $U1$ and $U2$ classes of PU. In this work we justify the thesis that the $Q2$ class covers a wide spectrum of exact solutions of time (space)-dependent quantum wave equations including Hulthen [11,12], Kratzer–Fues [19,20], Tipping [21] and a combination of Kratzer–Fues–Tipping potentials. It will be proved that the multiparametric generalization of the latter can be successfully applied in theoretical analysis of molecular spectra.

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2. Theory

In the extended CDG theory [1], the quantum $Q0$, $Q1$ and $Q2$ classes of PU can be obtained from the set of nonlinear equations [2]

$$\frac{d\psi(q)}{dq} - x(q)\psi(q) = 0, \qquad \frac{dx(q)}{dq} + \Phi(x) = 0 \tag{1}$$

in which $q = u_t t$ or $q = u_r r$ denote dimensionless temporal (spatial) variable, $u_{t(r)}$ is a scaling factor, whereas $\Phi(x)$ stands for a generating function, which expanded into a series of x-variable [1]

$$\Phi(x) = c_1(x + c_0/c_1) + c_2(x + c_0/c_1)^2 + \ldots \tag{2}$$

produces classical [2] and quantum [1] solutions

$$\psi(q) = \exp\left[-\int_x \frac{xdx}{\Phi(x)} + C\right] = \exp\left[\int_q x(q)dq + C\right] \tag{3}$$

for different powers $n = 0, 1, 2$ of the truncated series (2). The first-order CDG equations (1) can be converted to the second-order onc [1]

$$\frac{d^2\psi(q)}{dq^2} - \psi(q)\frac{dx(q)}{dq} - x(q)\frac{d\psi(q)}{dq} =$$
$$\left[-\frac{1}{2}\frac{d^2}{dq^2} + V(q) - \epsilon\right]\psi(q) = \left(\hat{H} - \epsilon\right)\psi(q) = 0,$$
$$V(q) - \epsilon = \frac{1}{2}\left[x(q)^2 + \frac{dx(q)}{dq}\right], \tag{4}$$

which includes the Riccati equation [22] associated with the potential energy $V(q)$ and eigenvalue ϵ characterizing the system under consideration. In this way one can generate the time (space)-dependent classical and quantum equations for different forms of generating function (2). For example, the constant term $\Phi(x) = c_0$ produces the quantum ground state wavefunction and associated eigenvalue for the harmonic oscillator, the linear expansion $\Phi(x) = c_0 + c_1 x$ gives exact solutions for the anharmonic Morse oscillator [4], whereas parabolic expansion $\Phi(x) = c_1(x + c_0/c_1) + c_2(x + c_0/c_1)^2$ yields solutions for the Wei oscillator [5], respectively [1]. In the dissociation limit, which corresponds to $c_0 = 0$, the quantum solutions reduce to the classical Gompertz [6] and West–Brown–Enquist (WBE)-type [7] functions belonging to $U1$ and $U2$ classes of PU.

The Hamilton operator appearing in Eq. (4) can be expressed in the terms of the first-order annihilation and creation operators $\hat{A}$ and $\hat{A}^\dagger$

$$\hat{H} = \frac{1}{\sqrt{2}}\left[-\frac{d}{dq} - x(q)\right]\frac{1}{\sqrt{2}}\left[\frac{d}{dq} - x(q)\right] = \hat{A}^\dagger \hat{A}, \tag{5}$$

hence, the CDG approach can be used to construct coherent states of harmonic and anharmonic oscillators which are eigenstates of the annihilation operator $\hat{A}$ [23,24]

$$\begin{aligned} &\hat{A}|\alpha\rangle = \alpha|\alpha\rangle, \\ &|\alpha\rangle = \psi(q)\exp[\sqrt{2}\alpha q], \quad [\hat{A}, \hat{A}^\dagger] = \Phi(x) \end{aligned} \tag{6}$$

and which minimize the generalized uncertainty relation ($\hbar = 1$) [23,25]

$$[\Delta x(q)]^2 (\Delta\epsilon)^2 \geq \frac{1}{4}\langle\alpha|\Phi(x)|\alpha\rangle^2, \; \Phi(x) = \mp i\left[x(q), \hat{\epsilon}\right]. \tag{7}$$

Here $\hat{\epsilon} = \pm i d/dq$ represents the energy (+) or momentum (−) operator whereas $x(q)$ plays the role of temporal (spatial) variable associated with an explicit form of the potential energy function $V(q)$. To obtain the minimum uncertainty coherent states we need only the ground state solution $\psi(q)$ and quantity $x(q)$ defining the annihilation operator $\hat{A}$.

3. Results

All quantities $x(q)$, $V(q)$, ϵ and $\psi(q)$ uniquely characterize the quantum systems and can be determined in the CDG scheme for different forms of the generating function $\Phi(x)$. It has been demonstrated [1] that only the form (2) produces quantal equations, whereas the conventional version $\Phi(x) = c_0 + c_1 x + c_2 x^2$ applied by CDG leads to the classical solutions for the dumped oscillations first time derived in [27]. A detailed mathematical analysis of the impact of $\Phi(x)$ on the form of quantum solutions generated in the CDG scheme revealed that they depend not only on different powers $n = 0, 1, 2$ of the truncated series (2) but also on the relations between parameters c_1, c_2 and the presence or absence the terms $(x + c_0/c_1)^l$ for $l < n$ in the series. Such modifications open the gate to obtain variety of new quantal solutions which are known or unknown in the domain of quantum physics.

3.1. Hulthen oscillator

Introducing the generating function (2) for $c_1 = c_2 = 1$

$$\Phi(x) = x + c_0 + (x + c_0)^2 \tag{8}$$

into the CDG equations (1), by integration one gets

$$x(q) = \frac{\exp(-q)}{1 - \exp(-q)} - c_0, \qquad \psi(q) = [1 - \exp(-q)]\exp(-c_0 q), \tag{9}$$

whereas the second-order eigenvalue equation (5) takes the form

$$\left[\frac{d^2}{dq^2} + \frac{(1 - 2c_0)\exp(-q)}{1 - \exp(-q)} - c_0^2\right]\psi(q) = 0 \tag{10}$$

identical as that obtained for the Hulthen oscillator [10–12]. The Hulthen model has been used in many branches of physics, such as nuclear [13], atomic [14,15], solid state [16] and chemical physics [17].

If we take into account the parameters relationships $c_0^2 = -\epsilon_1$ and $1 - 2c_0 = \beta^2$ one gets $\epsilon_1 = -[(\beta^2 - 1)/2]^2$, which represents the ground state eigenvalue included in the general formula [10]

$$\epsilon_v = -\left(\frac{\beta^2 - v^2}{2v}\right)^2 \qquad v = 1, 2, 3\ldots \tag{11}$$

associated with eigenfunctions (in arbitrary normalization) [10]

$$\begin{aligned} \psi(q)_v = {} & [1 - \exp(-q)]\exp(-c_0 q)\,{}_2F_1[2c_0 + 1 + v, 1 - v, \\ & 2c_0 + 1; \exp(-q)]. \end{aligned} \tag{12}$$

For $v = 1$, $\psi(q)$ specified by (9) takes identical form as the ground state solution $\psi(q)_1$ obtained from (12).

Having derived the ground state eigenfunction $\psi(q)_1$ and $x(q)$ appearing in the operator $\hat{A}$ defining by Eq. (5) one may construct the time- and space-dependent coherent states of the Hulthen oscillator employing the formula (6)

$$|\alpha\rangle = [1 - \exp(-q)]\exp(-c_0 q)\exp[\sqrt{2}\alpha q]. \tag{13}$$

They are eigenstates of the annihilation operator

$$\frac{1}{\sqrt{2}}\left[\frac{d}{dq} - \frac{\exp(-q)}{1 - \exp(-q)} + c_0\right]|\alpha\rangle = 0, \tag{14}$$

and minimize the generalized uncertainty relation (7)

$$[\Delta x(q)]^2 (\Delta\epsilon)^2 = \frac{1}{4}\langle\alpha|\Phi(x)|\alpha\rangle^2, \quad \Phi(x) = x + c_0 + (x + c_0)^2, \tag{15}$$

with variable $x(q)$ defined by (9). The minimum uncertainty coherent states (13) for the Hulthen oscillator have not been constructed yet and only ladder (raising and lowering) operators satisfying $SU(0)$ commutation relation were derived for this model [18].

3.2. Kratzer–Fues–Tipping oscillator

Lets consider the generating function $\Phi(x)$ including only the second-order term with parameter $c_2 = c_1$

$$\Phi(x) = c_1(x + c_0/c_1)^2. \tag{16}$$

Introducing (16) into (1) and carried out calculations with respect to the boundary condition $x(0) = (1 - c_0)/c_1$ one gets

$$x(q) = \frac{1}{c_1(1 + q)} - \frac{c_0}{c_1}, \quad \psi(q) = (1 + q)^{1/c_1}\exp\left(-\frac{c_0}{c_1}q\right) \tag{17}$$

and the second-order eigenvalue equation (5)

$$\left\{-\frac{1}{2}\frac{d^2}{dq^2} + D\left[\frac{(1 + q) - k}{1 + q}\right]^2 - \epsilon\right\}\psi(q) = 0 \tag{18}$$

in which

$$D = \frac{c_0^2}{c_1^2(1 - c_1)}, \qquad \epsilon = D - \frac{c_0^2}{c_1^2}, \qquad k = \frac{1 - c_1}{c_0}. \tag{19}$$