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Coarse analysis of collective behaviors: Bifurcation analysis of the optimal velocity model for traffic jam formation

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ABSTRACT

We present a general method for analyzing macroscopic collective phenomena observed in many-body systems. For this purpose, we employ diffusion maps, which are one of the dimensionality-reduction techniques, and systematically define a few relevant coarse-grained variables for describing macroscopic phenomena. The time evolution of macroscopic behavior is described as a trajectory in the low-dimensional space constructed by these coarse variables. We apply this method to the analysis of the traffic model, called the optimal velocity model, and reveal a bifurcation structure, which features a transition to the emergence of a moving cluster as a traffic jam.

1. Introduction

Collective phenomena have been investigated in various fields such as materials sciences, biological sciences, and social sciences. Examples of such phenomena include granular flow [1–4], the collective motion of organisms [5–7], traffic flow [8–10,3,4], and pedestrian flow and evacuation dynamics [11–13]. These phenomena emerge as the time evolution of the macroscopic aspects of many-body systems. Collective phenomena show the general properties of dynamical deformation and transformation, and they sometimes appear as drastic global changes. Many physicists have recently shown increasing interest in the investigation of such phenomena. From a physical viewpoint, these phenomena can generally be considered as the collective motion of individual elements.

When we focus on the macroscopic aspects of collective phenomena, it is convenient to express the states of these phenomena using a few appropriate macroscopic variables, like thermodynamic variables and thermodynamic potentials such as free energy. Here, we take group formation as an example of collective phenomena. The process of group deformation can be considered to be a change in the similarity between patterns at each sequential time. We require coarse-grained variables to construct a low-dimensional metric space in which the similarity between patterns is measured. These variables are obtained by using pattern classification techniques. In particular, in our study, we employ the method of diffusion maps [14–17]. By using this method, we obtain a few relevant coarse-grained variables that identify the differences among macroscopic patterns, and we use these as macroscopic variables. Then, we construct a macroscopic theory for non-equilibrium collective phenomena. The time development of the macroscopic state of the group is represented as a trajectory in the low-dimensional space composed of the obtained variables.

In this study, we construct the macroscopic theory by using diffusion maps for analyzing the macroscopic property of a non-equilibrium dissipative system. As an example of the macroscopic phenomena of collective particle motion, we focus on the phenomena of cluster formation such as jam formation. We analyze these phenomena using the optimal velocity model [18]. The model is investigated as a typical non-equilibrium dissipative system with asymmetric interaction, which violates the action-reaction principle. Jam formation is considered the dynamical phase transition of a non-equilibrium system [19–22]. One important property of the transition is the bistability of jammed flow and free flow. This property has been investigated in many studies using the optimal velocity model [23–27]; however, this property has not been satisfactorily explained. The microscopic equation of motion of each particle and the non-linearity of the interaction make it difficult to investigate this property in detail. Thus, we propose a new method using diffusion maps for analyzing the macroscopic property of jam formation using the optimal velocity model.

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The remainder of this paper is organized as follows. In section 2, we introduce the method of diffusion maps. In section 3, we provide a brief review of the optimal velocity model. In section 4, we apply diffusion maps to the optimal velocity model and find the bifurcation structure. Finally, section 5 presents the summary and discussion.

2. Diffusion maps

Diffusion maps [14–17] are one of the dimensionality-reduction techniques originally proposed in the field of data analysis. In this study, we use this technique to obtain coarse variables for describing macroscopic phenomena in systems with many degrees of freedom. This technique results in a transformation of variables (called a *diffusion map*) from a large number of microscopic variables to a few coarse macroscopic variables. In this section, we give a brief explanation of diffusion maps.

The procedure for constructing a diffusion map follows. First, we consider a data set $X = \{\boldsymbol{x}_i \in \mathbb{R}^N \mid i = 1 \ldots, M\}$. In analysis of the collective dynamics, each data point $\boldsymbol{x}_i$ is an N-dimensional vector representing a state of a system with N degrees of freedom. For example, in a classical mechanics system, the components of $\boldsymbol{x}_i$ are the positions and velocities of all particles at time t_i. Here, we note that it is not necessary to order the indices of states $\{\boldsymbol{x}_i\}$ temporally, and a data set is allowed to contain time-sequential data obtained from different initial conditions. In the method of diffusion maps, only the similarities between states are essential.

Next, we define a Markov chain, which describes the transits from one state to another. The probability of a transition between states $\boldsymbol{x}_i$ and $\boldsymbol{x}_j$ is defined as

$$p(\boldsymbol{x}_i, \boldsymbol{x}_j) := \frac{\exp(-\frac{\|\boldsymbol{x}_i - \boldsymbol{x}_j\|^2}{\varepsilon^2})}{\sum_{j'=1}^{M} \exp(-\frac{\|\boldsymbol{x}_i - \boldsymbol{x}_{j'}\|^2}{\varepsilon^2})}, \tag{1}$$

where $\|\cdot\|$ is the Euclidean norm, and the denominator is the normalization constant. Here, ε is a scaling parameter, which should be given a sufficiently small value so that a state can transit only within its neighborhood by one step. Then, the Markov chain can be thought of as a *diffusion* process for the set of states X.

Moreover, we introduce a distance function between states $\boldsymbol{x}_i$ and $\boldsymbol{x}_j$ as

$$D_s^2(\boldsymbol{x}_i, \boldsymbol{x}_j) := \sum_{m=1}^{M} \frac{(p_s(\boldsymbol{x}_i, \boldsymbol{x}_m) - p_s(\boldsymbol{x}_j, \boldsymbol{x}_m))^2}{\phi_0(\boldsymbol{x}_m)}. \tag{2}$$

Here, D_s is called the *diffusion distance*, where $p_s(\boldsymbol{x}_i, \boldsymbol{x}_m)$ is the transition probability from $\boldsymbol{x}_i$ to $\boldsymbol{x}_m$ in s steps, and ϕ_0 is the stationary distribution function of the Markov chain. $D_s(\boldsymbol{x}_i, \boldsymbol{x}_j)$ can be regarded as the measure of similarity between the distributions $p_s(\boldsymbol{x}_i, \cdot)$ and $p_s(\boldsymbol{x}_j, \cdot)$. The right-hand side of equation (2) can be rewritten as follows. We define the Markov transition matrix P whose (i, j) element is

$$P_{ij} = p(\boldsymbol{x}_i, \boldsymbol{x}_j), \tag{3}$$

and its eigenvalue λ_α and corresponding right eigenvector $\boldsymbol{\psi}_\alpha$ satisfy

$$P\boldsymbol{\psi}_\alpha = \lambda_\alpha \boldsymbol{\psi}_\alpha, \quad (\alpha = 0, 1, \ldots, M-1). \tag{4}$$

Owing to the properties of a Markov transition matrix, the eigenvalues can be ordered as

$$1 = \lambda_0 \geq \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{M-1} \geq 0, \tag{5}$$

without loss of generality. Using $p_s(\boldsymbol{x}_i, \boldsymbol{x}_j) = ((P)^s)_{ij}$ and the spectral decomposition of $(P)^s$, we can transform equation (2) into

$$D_s^2(\boldsymbol{x}_i, \boldsymbol{x}_j) = \sum_{\alpha=1}^{M-1} ((\lambda_\alpha)^s \psi_\alpha(\boldsymbol{x}_i) - (\lambda_\alpha)^s \psi_\alpha(\boldsymbol{x}_j))^2, \tag{6}$$

where $\psi_\alpha(\boldsymbol{x}_i)$ is the i-th component of the right eigenvector $\boldsymbol{\psi}_\alpha$.

Then, we define the coordinate transformation $\boldsymbol{y}_s : \mathbb{R}^N \to \mathbb{R}^{M-1}$ as

$$\begin{aligned} \boldsymbol{x}_i \mapsto \boldsymbol{y}_s(\boldsymbol{x}_i) &= \big(y_{s,1}(\boldsymbol{x}_i), \ldots, y_{s,M-1}(\boldsymbol{x}_i)\big) \\ &= \big((\lambda_1)^s \psi_1(\boldsymbol{x}_i), \ldots, (\lambda_{M-1})^s \psi_{M-1}(\boldsymbol{x}_i)\big). \end{aligned} \tag{7}$$

The coordinate transformation map $\boldsymbol{y}_s$ is called a diffusion map. In the space that consists of diffusion map coordinates, the diffusion distance (i.e., equation (6)) equals the Euclidean distance,

$$\begin{aligned} D_s^2(\boldsymbol{x}_i, \boldsymbol{x}_j) &= \sum_{\alpha=1}^{M-1} (y_{\alpha,s}(\boldsymbol{x}_i) - y_{\alpha,s}(\boldsymbol{x}_j))^2 \\ &= \|\boldsymbol{y}_s(\boldsymbol{x}_i) - \boldsymbol{y}_s(\boldsymbol{x}_j)\|^2. \end{aligned} \tag{8}$$

Now, we are ready to perform dimensionality reduction. Using the inequality (5), the right-hand side of equation (8) can be approximated as

$$\|\boldsymbol{y}_s(\boldsymbol{x}_i) - \boldsymbol{y}_s(\boldsymbol{x}_j)\|^2 \cong \sum_{\alpha=1}^{n} (y_{s,\alpha}(\boldsymbol{x}_i) - y_{s,\alpha}(\boldsymbol{x}_j))^2, \tag{9}$$

for appropriately large values of n and s. The approximate equation (9) means that the similarity between two states can be coarsely characterized using up to n coordinates $(y_{1,s}, \ldots, y_{n,s})$. In our study, these n variables are the coarse variables used to describe collective dynamics. The variables n and s can be assumed to be small values based on the properties of macroscopic structures. In Sec. 4, we explore the case of $n = 2$, $s = 1$. We use only two variables, $\lambda_1\psi_1, \lambda_2\psi_2$, to describe the macroscopic phenomena of the optimal velocity model.

We wish to study the variation of macroscopic properties under different control parameters and conditions using coarse variables in the same reduced space. For this purpose, we use the transformation of an arbitrary state $\tilde{\boldsymbol{x}}$, which has N components, to the space of diffusion map coordinates obtained by the original data set X. Note that $\tilde{\boldsymbol{x}}$ is not necessarily included in X. By using *the Nyström extension* [28,29], such a transformation is represented by

$$y_{s,\alpha}(\tilde{\boldsymbol{x}}) = \sum_{j=1}^{M} p_s(\tilde{\boldsymbol{x}}, \boldsymbol{x}_j)\psi_\alpha(\boldsymbol{x}_j), \quad (\alpha = 1, \ldots, M-1), \tag{10}$$

to keep it consistent with equation (7). According to equation (10), we take the variables for $\alpha = 1, \ldots, n$. Then, we can analyze the dynamics using coarse variables $(y_{s,1}, \ldots, y_{s,n})$.

3. The optimal velocity model

In this section, we briefly review the optimal velocity model [18], used as an example of diffusion map analysis. The optimal velocity model was originally proposed as a model for traffic flow to explain the emergence of traffic jams. The model is formulated using an equation of motion for each particle:

$$\begin{aligned} &\frac{d^2x_i(t)}{dt^2} = a\Big(\mathrm{V}(\Delta x_i(t)) - \frac{dx_i(t)}{dt}\Big), \\ &\Delta x_i := x_{i+1}(t) - x_i(t), \end{aligned} \tag{11}$$

where $x_i(t)$ $(i = 1, \ldots, N)$ is the position of the i-th particle in a one-dimensional lane at time t. Here, a is a constant representing the sensitivity of a particle. $\mathrm{V}(\Delta x_i)$ is called the optimal velocity (OV) function, which is a sigmoidal function and represents the