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General and exact pressure evolution equation

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ABSTRACT

A crucial issue in fluid dynamics is related to the knowledge of the fluid pressure. A new general pressure equation is derived from compressible Navier–Stokes equation. This new pressure equation is valid for all real dense fluids for which the pressure tensor is isotropic. It is argued that this new pressure equation allows unifying compressible, low-Mach and incompressible approaches. Moreover, this equation should be able to replace the Poisson equation in isothermal incompressible fluids. For computational fluid dynamics, it can be seen as an alternative to Lattice Boltzmann methods and as the physical justification of artificial compressibility.

1. Introduction

Incompressible Navier–Stokes equations (INS) describes a fluid characterized by infinite sound speed. It is valid in the case of fluid flows in isothermal configuration and at low Mach numbers (Mach number, $Ma = U^\star/c^\star$ is the ratio of the characteristic flow speed $U^\star$ and the speed of sound $c^\star$ defined at some reference temperature $T^\star$ and density $\rho^\star$). INS equations correspond to a mixture of hyperbolic and elliptic partial differential equations. They can be written

$$\partial_t u_i + u_j \partial_j u_i + \partial_i P = \frac{1}{Re} \partial_j \partial_j u_i \quad \partial_i u_i = 0 \tag{1}$$

where $\mathbf{u}$ is the fluid velocity, P is the pressure and Re the Reynolds number, which represents the ratio between inertial and viscous forces [1]. The pressure in (1) is not an independent thermodynamic variable. It can be seen as a Lagrangian multiplier of the incompressibility constraint. It is determined by the Laplace or Poisson equation:

$$\partial_i \partial_i P = -(\partial_j u_i)(\partial_i u_j) \tag{2}$$

In very anisothermal flow, the low Mach number hypothesis conducts to a similar system [2,3]. Considering that only density ρ depends on temperature, the low Mach number equations can be written

$$\begin{aligned} \partial_t u_i + u_j \partial_j u_i + \frac{1}{\rho}\partial_i P &= \frac{1}{\rho Re}\left(\partial_j \partial_j u_i + \frac{1}{3}\partial_i S\right) \\ \partial_i u_i &= S \end{aligned} \tag{3}$$

where ρ depends on temperature and S is a source term linked to conductive heat transfer (S depends on temperature). Again, the pressure is determined by a Poisson equation. It can be given by:

$$\begin{aligned} \partial_i\left(\frac{1}{\rho}\partial_i P\right) = {} & \frac{1}{Re}\frac{4}{3\rho}\partial_j\partial_j S + \\ & \frac{1}{Re}\partial_j\left(\frac{1}{\rho}\right)\left(\partial_i\partial_i u_j + \frac{1}{3}\partial_i S\right) \\ & -(\partial_j u_i)(\partial_i u_j) - u_j \partial_j S - \partial_t S \end{aligned} \tag{4}$$

The physical meaning of (2) and (4) is that in a system with infinitely fast sound propagation, any pressure disturbance induced by the flow is instantaneously propagated into the whole domain. This elliptic problem is a crucial issue for fluid dynamics. Indeed, the INS equations are difficult to study analytically and numerically. This difficulty has motivated the search of alternative numerical approaches to determine pressure without solving the Poisson equation. Three different ways have been found. The first is the so-called artificial compressibility method where a pressure evolution equation is postulated [4]. The second way is the Lattice Boltzmann method (LBM) which uses a velocity-space truncation of the Boltzmann equation from the kinetic theory of gases [5]. The third way consists in adopting an inverse kinetic theory which permits the identification of the (Navier–Stokes) dynamical system and

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of the corresponding evolution operator which advances in time the kinetic distribution function and the related fluid fields [6]. The pressure evolution equation obtained by this method is non-asymptotic. The full validity of INS equations is preserved.

In this paper, we determine a general and exact pressure evolution equation for all real dense fluids for which the pressure tensor is isotropic. Unlike the work of Tessarotto et al. [6], the obtained pressure equation is a physical one and not a mathematically rigorous theory for INS equations. The obtained general and exact pressure evolution equation gives the physical bases of artificial compressibility method and it allows the study of very anisothermal flow contrary to LBM. The goal is similar to the reduced compressible Navier–Stokes equations (RCNS) derived by Ansumali et al. [7] and the proposed pressure equation is very similar to the grand potential equation derived by Karlin et al. [8]. However we will argue that the use of pressure instead of the grand potential simplifies the equation of compressible hydrodynamics. Moreover, because the proposed pressure equation is valid for all real dense fluids, it builds bridge between compressible, low-Mach and incompressible approaches.

In section 2, we will determine the general and exact pressure evolution equation (without any additional assumptions). This equation generalizes the one used by Zang et al. in the particular case of an ideal gas [9]. In section 3, we will simplify this equation in the low Mach number limit. Finally, in section 4, we will reduce the equation for low Mach number and isothermal flow.

2. General pressure evolution equation

The total energy E conservation is given by

$$\partial_t(\rho E) + \partial_j((\rho E + p)u_j) = \partial_j(\sigma_{ij}u_i) - \partial_j q_j \tag{5}$$

with σ_{ij} the shear-stress tensor for a Newtonian fluid

$$\sigma_{ij} = 2\mu S_{ij} - \frac{2}{3}\mu\delta_{ij}S_{kk} \quad S_{ij} = \frac{1}{2}(\partial_i u_j + \partial_j u_i) \tag{6}$$

and q_i the conductive heat flux

$$q_i = -\kappa\partial_i T \tag{7}$$

Introducing internal energy $U = E - \frac{u_i u_i}{2}$ and enthalpy $H = U + \frac{P}{\rho}$, one gets

$$\rho D_t H = D_t P - \partial_i q_i + \Phi \qquad \Phi = \sigma_{ij}\partial_i u_j \tag{8}$$

with D_t the material derivative (total derivative [1]):

$$D_t\phi = \partial_t\phi + u_i\partial_i\phi \tag{9}$$

Using the relations of heat capacity at constant pressure c_p and of the isobaric thermal expansion coefficient α

$$\left(\frac{\partial H}{\partial T}\right)_P = c_p$$
$$\left(\frac{\partial H}{\partial P}\right)_T = \frac{1}{\rho}(1 - T\alpha) \text{ with } \alpha = -\frac{1}{\rho}\left(\frac{\partial\rho}{\partial T}\right)_P \tag{10}$$

an alternative formulation can be derived for temperature

$$\rho c_p D_t T = T\alpha D_t P - \partial_i q_i + \Phi \tag{11}$$

We propose to derive a new pressure equation from the temperature formulation (11). We introduce the isothermal compressibility coefficient $\chi_T = \frac{1}{\rho}\left(\frac{\partial\rho}{\partial P}\right)_T$ and we consider temperature as a function of density and pressure $D_t T = \left(\frac{\partial T}{\partial\rho}\right)_P D_t\rho + \left(\frac{\partial T}{\partial P}\right)_\rho D_t P$. Using mass conservation and (11), the recomputation poses no difficulties and we here write the result:

$$(\rho c_p\frac{\chi_T}{\alpha} - \alpha T)D_t P = -\partial_i q_i - \frac{\rho c_p}{\alpha}\partial_i u_i + \Phi \tag{12}$$

In order to simplify this expression, one introduces isochoric heat capacity $c_v = \left(\frac{\partial U}{\partial T}\right)_\rho$, heat capacity ratio $\gamma = \frac{c_p}{c_v}$ and the Mayer relation

$$\alpha^2 T = \rho c_v \chi_T(\gamma - 1) \tag{13}$$

One obtains

$$D_t P + \frac{\gamma}{\chi_T}\partial_i u_i = \frac{\alpha}{\rho c_v\chi_T}(\Phi - \partial_i q_i) \tag{14}$$

Sound velocity c and the isentropic compressibility coefficient χ_S are given by

$$c^2 = \left(\frac{\partial P}{\partial\rho}\right)_S$$
$$\chi_S = \frac{1}{\rho}\left(\frac{\partial\rho}{\partial P}\right)_S \tag{15}$$

Using the Reech relation $\gamma = \frac{c_p}{c_v} = \frac{\chi_T}{\chi_S}$, one obtains the general and exact pressure evolution equation

$$D_t P + \rho c^2\partial_i u_i = \frac{\alpha}{\rho c_v\chi_T}(\Phi - \partial_i q_i) \tag{16}$$

In the particular case of an ideal gas $\alpha = \frac{1}{T}$, $\chi_T = \frac{1}{P}$ and $c^2 = \gamma r T$ with r the specific gas constant, this equation is equivalent to the one used by Zang et al. [9,10]. It is worth noting that equation (16) can be used for any real dense fluids (gas or liquid) without restriction on Mach number or temperature gradient. It gives the physical bases of artificial compressibility methods that postulate the pressure equation. The pressure equation (16) can be seen as an energy equation: to complete the system, one has to consider in addition mass conservation, momentum conservation and an equation of state.

3. Pressure equation for low Mach number flow

At this step, the pressure evolution equation (16) depends on the total derivative. At low Mach number, the first simplification consists in assuming that viscous dissipation is negligible. We now show that, at low Mach number, advection can be neglected. One defines the following nondimensionalized quantities:

$$\rho^X = \frac{\rho}{\rho^\star} \qquad u_i^X = \frac{u_i}{c^\star}$$
$$P^X = \frac{\gamma P}{\rho^\star(c^\star)^2} \qquad t_P^X = \frac{1}{Ma^2}\frac{tU^\star}{x^\star} \tag{17}$$

It is worth noting that in the classical low Mach number assumption, nondimensionalized time is defined by $t_U^X = \frac{tU^\star}{x^\star}$. The factor $\frac{1}{Ma^2}$ is justified by the fact that pressure time evolution is much faster than velocity time evolution (subscripts $_U$ and $_P$ indicate that the nondimensionalized time corresponds to velocity or pressure respectively). One defines moreover, the Reynolds number Re, the Prandtl number Pr and the Peclet number Pe:

$$Re = \frac{\rho U^\star x^\star}{\mu} \quad Pr = \frac{\nu}{a_T} \quad Pe = PrRe \tag{18}$$

One uses the asymptotic expansion of pressure, temperature and velocity

$$P^X = P_0 + Ma^2 P_1 \tag{19}$$
$$T^X = T_0 + Ma^2 T_1 \tag{20}$$
$$u_i^X = Ma(u_{i0} + Ma^2 u_{i1}) \tag{21}$$