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Effects of heterogeneity in site–site couplings for tight-binding models on scale-invariant structures

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ABSTRACT

We studied the thermodynamic behaviors of non-interacting bosons and fermions trapped by a scale-invariant branching structure of adjustable degree of heterogeneity. The full energy spectrum in tight-binding approximation was analytically solved. We found that the log-periodic oscillation of the specific heat for Fermi gas depended on the heterogeneity of hopping. Also, low dimensional Bose–Einstein condensation occurred only for non-homogeneous setup.

1. Introduction

Tight-binding quantum gases upon quasiperiodic or fractal-like structures with scale symmetry have been studied intensively over the past few decades [1–23]. In most cases, the energy spectrum of the ideal gas and corresponding density of states show self-similarity and power-law behaviors at the same time. This is responsible for a sequence of unique behaviors related to localization of states [1–3], quantum transport [4–6], specific heat [7–14], Bose–Einstein condensation (BEC) [15–23], etc. Though without introducing interaction, the simplest model yields lots of interesting phenomena due to the complex topology of fractal-like lattice structures. Different from isotropic models, the hopping of particles is non-trivial in these cases. Naturally one will ask how the heterogeneity of hopping (site–site coupling) influences the model.

The heterogeneity of hopping consists of two aspects: the network topology of lattices and the variation of coupling strength. There have been many results on how the topology of lattice structures gives birth to unusual behaviors of hopping gases. For example, the low dimensional BEC of non-interacting bosons, trapped by diamond hierarchical lattices, only takes place while the branching parameter of the trap structure is lager than 2 [23]. Recently, the quantum transport on Sierpinski carpets is also found to be determined by structural parameters [6]. One can guess the topological properties of lattice structures decide how curved the underlying space is for the hopping gas. Though locally similar to an isotropic Euclidean lattice, a fractal-like structure can produce totally different outcomes when serving as traps for hopping gas. To describe those anisotropic structures more quantitatively, some indicators including the fractal dimension and the spectral dimension [24–28] are introduced. A deterministic relation among them is also provided for some renormalizable structures [29].

However, it is rarely reported that how the heterogeneity of the strength of site–site couplings (hopping amplitude) influences the behaviors of quantum gases. The heterogeneity of hopping amplitudes is worth studying since the site–site coupling is suggested to play an important role in other similar models. There are many cases that can not be approached by mean field approximation in real world systems. For example, the heterogeneity in the site–site coupling significantly affects the epidemic spreading [30,31], transportation [32,33], synchronization [34,35], random walks [36,37], diffusive processes [38], voter models [39,40], etc., on weighted networks. We will fill this gap by a case study regarding the non-interacting Fermi and Bose gases upon a parameterized scale-invariant branching structure. We will show that the heterogeneity of coupling strength has a decisive influence on the thermodynamic behaviors even in the simplest model.

This paper is organized as the following. First we construct a scale-invariant branching structure with two parameters control-

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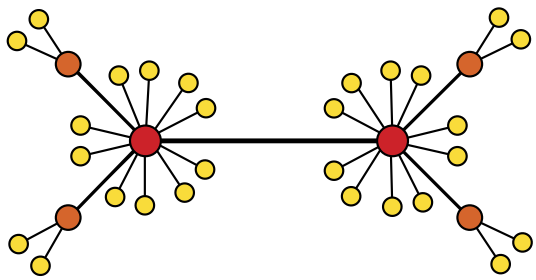

Fig. 1. A sketch of $G(2,1)$.

ling the heterogeneity of our model. In tight-binding approximation we define the normalized Hamiltonian. By appropriate renormalization the full spectrum is obtained. Then, for Fermi gas, we study its Fermi energy and subsequent log-periodic oscillation of specific heat associated to special weight parameter. As for Bose gas, we investigate its phase transition phenomenon at low temperature and find the relation between the weight parameter and BEC.

2. Preliminary

A weighted branching structure is constructed iteratively, see Fig. 1.

$G^{(0)}$ is a chain of length 1 where two vertices are connected by an edge of unit weight. For $t>0$, $G^{(t)}$ is obtained from $G^{(t-1)}$ by the following transformation. For each edge of weight w in $G^{(t-1)}$, $mw(m>0)$ new vertices are connected to both sides of the edge respectively with unit weight, meanwhile, the weight of the old edge is increased by $m\theta w(\theta \geqslant 0)$. The parameters m,θ are all integers. Let $G(m,\theta)=\lim_{t\to\infty} G^{(t)}$. An infinite branching structure forms.

By construction, the total number of the vertices for $G^{(t)}$ is

$$N_t = \frac{2}{2+\theta}[(\theta m+2m+1)^t+\theta+1]. \tag{1}$$

Name these vertices by $v_1, v_2, \cdots, v_{N_t}$. a_{ij} denotes the weight of the edge connecting v_i and v_j. a_{ij} is 0 when v_i and v_j are not adjacent. Further we define the degree of v_i as $d_i=\sum_j a_{ij}$.

To describe the topological structure of $G^{(t)}$, we introduce the adjacency matrix $(A^{(t)})_{ij}=a_{ij}$ and the degree matrix $(D^{(t)})_{ij}=\delta_{ij}d_i$. Let the normalized stochastic matrix [41] for Markov chains on $G^{(t)}$ be $T^{(t)}=\sqrt{D^{(t)}}A^{(t)}\sqrt{D^{(t)}}$. Obviously, $t_{ij}=\frac{a_{ij}}{\sqrt{d_i d_j}}$. For $G(m,\theta)$, define $T=\lim_{t\to\infty} T^{(t)}$.

3. Tight-binding model on $G(m,\theta)$

Suppose the structure we constructed denotes a trapping structure for quantum gases. The edges connecting two vertices represent the correlation of two traps. The tight-binding Hamiltonian describing the system writes [18,21]

$$\hat{\mathcal{H}}_0 = \sum_i d_i \hat{a}_i^\dagger \hat{a}_i - \sum_{ij} a_{ij}\hat{a}_i^\dagger \hat{a}_j. \tag{2}$$

Here $\hat{a}_i^\dagger$ and $\hat{a}_i$ are creation and annihilation operators and a_{ij} denotes the hopping amplitude between two coupled traps. The second summation in Eq. (2) is taken over all neighboring vertices i and j. Clearly, when $\theta=0$, a_{ij} is constantly 1 for all existing site–site correlations. This is the most homogeneous case in our model. For non-vanishing θ, the hopping amplitude is heterogeneous.

From Eq. (2), we know the spectrum of $\hat{\mathcal{H}}_0$ is unbound for infinite network ($t\to\infty$). However, by rescaling the frequency space (multiplying the Hamiltonian by diagonal operators at both sides), we can normalize $\hat{\mathcal{H}}_0$ as

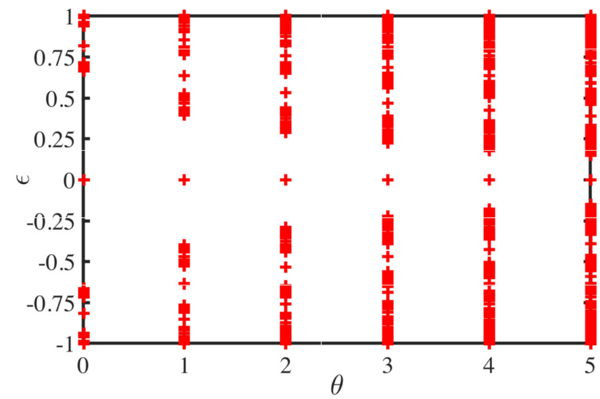

Fig. 2. (Color online.) Eigenvalue spectra related to θ from 0 to 6 when $m=2$.

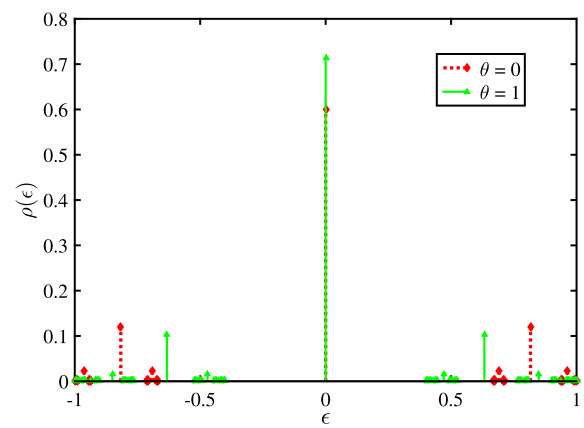

Fig. 3. (Color online.) Density of states associated to $G(2,0)$ and $G(2,1)$.

$$\hat{\mathcal{H}} = -\sum_{ij} t_{ij}\hat{a}_i^\dagger \hat{a}_j, \tag{3}$$

of which the spectrum lies on $[-1,1]$.

The matrix T we defined previously hence gives a full description of $\hat{\mathcal{H}}$. The allowed energy for $\hat{\mathcal{H}}$ is the eigenvalue spectrum of $-T$. The spectrum is a Julia multiset $J_R \subset [-1,1]$ generated by the inverse of the function

$$R(x) = \frac{\theta m+m+1}{\theta m+1}x - \frac{m}{(\theta m+1)x} \tag{4}$$

from $\{-1,1\}\bigcup\{0\}$ [42,43]. A detailed description of J_R is provided in the appendix. Fig. 2 shows how the eigenvalue spectra vary with the weight parameter θ. When $\theta\to\infty$, the spectrum is dense in $[-1,1]$.

Let $\deg^{(t)}(\varepsilon)$ denote the multiplicity of the eigenvalue ε of $-T^{(t)}$. Take $\deg^{(t)}(\varepsilon)=0$ if ε is not an eigenvalue. The density of states on $[-1,1]$ is

$$\rho(\varepsilon) = \sum_{\varepsilon'\in J_R} \delta(\varepsilon-\varepsilon') \lim_{t\to\infty} \frac{\deg^{(t)}(\varepsilon')}{N_t} \tag{5}$$

where $\delta(\varepsilon-\varepsilon')$ is the Dirac delta function. Fig. 3 is a schematic representation of $\rho(\epsilon)$ related to different θ. Obviously, $\rho(\epsilon)$ shows self-similar properties.

Fig. 2 and Fig. 3 together show that the spectrum related to $G(m,\theta)$ is highly degenerate and fractal-like. Besides, the spectrum is symmetric with respect to $\epsilon=0$, which possesses the largest