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Localization and mobility edges in the off-diagonal quasiperiodic model with slowly varying potentials

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ABSTRACT

We study a one-dimensional system that includes both a commensurate off-diagonal modulation of the hopping amplitude and an incommensurate, slowly varying diagonal on-site modulation. By using asymptotic heuristic arguments, we identify four closed form expressions for the mobility edges. We further study numerically the inverse participation ratio, the density of states and the Lyapunov exponent. The numerical results are in exact agreement with our theoretical predictions. Besides a metal–insulator transition driven by the strength of the slowly varying potential, another four insulator–metal transitions are found in this model as the energy is increased in magnitude from the band center ($E = 0$) to the mobility edges ($\pm E_{c2}, \pm E_{c1}$).

1. Introduction

Anderson localization [1], the absence of diffusion of matter waves in disorder systems, is an active research subject in condensed matter physics due to its importance and complexities. With regard to the randomly-distributed potentials the scaling theory [2] predicts that there is no metal–insulator transition in one-dimensional (1D) systems. Hence all wave-functions are exponentially localized no matter how small the existed disorder is. However, the emergence of quasiperiodic/incommensurate potentials in 1D systems [3–13] goes beyond this prediction. As an important paradigm, the Aubry–André (AA) model [14] can undergo a transition from the extended state to the localized state as the amplitude of the incommensurate potential increases. The nature of the AA model has been well understood with extensive researches [15–20]. It is well known that at the phase transition point the spectrum of the AA model is a Cantor set and all wave-functions are critical, i.e., neither extended nor localized. However, unlike the Anderson transition in 3D situations, there is no mobility edge [21–23] in the AA model, where all wave-functions are either extended or localized (critical), depending on the relative strength of the incommensurate potential and the hopping amplitude.

The concept of the mobility edge, first proposed by Mott [24] regarding the 3D Anderson model, is signaled by an energy-level border separating the localized and extended bands. Mott argued, if the extended and localized states can coexist at the same spectral energy level, then, an arbitrarily small energy perturbation leads to the hybrid between localized and extended states. As a result, only extended states can remain, which is not true. However, whether a 1D system can host mobility edges is another interesting question. About thirty years ago a unique class of systems with 1D potentials [25–29], i.e., very slowly varying incommensurate potentials in real space, was introduced and extensively studied. These deterministic (diagonal) potentials are neither random nor simply incommensurate. The important finding by Sarma et al. [28] is that, there is a metal–insulator transition in this model with the mobility edges located at certain energies with the eigenstates at the band center being all extended whereas the band-edge states all localized. Since then, different variations of the 1D models containing mobility edges have been studied, including those with a rational term added to the slowly varying incommensurate potential [29] or the Aubry–André-like model with a duality symmetry [30,31]. Recently, the off-diagonal AA model, an extension of the AA model including both commensurate and incommensurate off-diagonal hopping modulations, has attracted much attention due to the arising rich and novel quantum phases, especially the important nontrivial zero-energy topological edge states [32–34].

Here we include an off-diagonal commensurate modulation in Sarma's model with slowly varying incommensurate diagonal modulations. While the latter brings out a metal–insulator transition, we find that, the former modulate the mobility edges greatly. This

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result opens perspectives for further investigations on the interplay between mobility edge and off-diagonal modulations.

Our generalized AA model, including commensurate off-diagonal and slowly varying incommensurate diagonal modulations, is expressed as

$$\hat{H} = -\sum_{i=1}^{L-1}(t+\lambda_i)(\hat{c}_i^{\dagger}\hat{c}_{i+1}+h.c.) + \sum_{i=1}^{L} V_i \hat{n}_i, \tag{1}$$

where $\hat{c}_i$ is the fermionic annihilation operator, $\hat{n}_i = \hat{c}_i^{\dagger}\hat{c}_i$ is the particle number operator, L is the total number of sites, $\lambda_i = \lambda\cos(2\pi bi)$ with $0<\lambda<1$ being the strength of the commensurate modulation on the off-diagonal hopping amplitude, and $V_i = V\cos(2\pi\beta i^{\nu}+\phi)$ with $0<\nu<1$ and $V>0$ being the strength of the slowly varying incommensurate potential. Without loss of generality, we choose the parameters $b=1/2$, $\beta=(\sqrt{5}-1)/2$, the phase in the incommensurate modulation $\phi=0$ and $\nu=0.5$. For convenience, $t=1$ is set as the energy unit.

When $\lambda=0$ and $\nu=1$, our model reduces to the AA model, while $\lambda=0$ and $0<\nu<1$ is the extended AA model studied by Sarma et al. [28]. It is well known that this model has two mobility edges located at energies $\pm|2-V|$. When $V<2$, all wave-functions with eigenenergies in $[V-2, 2-V]$ are extended and otherwise localized. When $V>2$, all wave-functions are localized, which is same as that in the AA model. In this paper, we focus on the situation with $0<\lambda<1$ and $0<\nu<1$. The main findings are, (i) when $V<1-\lambda$ there exist two additional mobility edges located at $\pm|2\lambda+V|$ besides those at $\pm|2-V|$, (ii) the all-wave-function-localized transition point is located at $V=1-\lambda$.

The rest of the paper is organized as follows. In Sec. 2, we theoretically give the heuristic arguments for the Hamiltonian (1). In Sec. 3, we present our numerical results and compare them with the theoretical analysis. The conclusion is summarized in Sec. 4.

2. Heuristic arguments

The slowly varying incommensurate potential brings up new nature of localization, which is substantially different from the 3D Anderson model [28]. To demonstrate the existence of mobility edges, Sarma et al. [28] presented some preliminary heuristic arguments and applied the semiclassical WKB technique. Here we also implement this method to obtain the explicit expressions of mobility edges.

By noticing that the slowly varying potential difference of $V_i = V\cos(2\pi\beta i^{\nu})$ vanishes in the thermodynamic limit [28], we thus write

$$\frac{dV_i}{di} = -2V\pi\beta i^{\nu-1}\sin(2\pi\beta i^{\nu}). \tag{2}$$

When $i\to\infty$, Eq. (2) can be written as

$$\lim_{i\to\infty}\left|\frac{dV_i}{di}\right| = -\lim_{i\to\infty} 2V\pi\beta\frac{|\sin(2\pi\beta i^{\nu})|}{i^{1-\nu}} = 0, \tag{3}$$

since $0<\nu<1$. Equivalently, the potential difference $V_{i+1}-V_i\to 0$ when the lattice number i is large enough, i.e., the potential V_i becomes a constant. This asymptotic property of "being constant" of V_i is crucial for the localization property of this model. On the other hand, another unique characteristic of our model is the existence of the off-diagonal commensurate modulation λ_i which leads to a parity symmetry. The Schrödinger equations for Eq. (1) rewritten according to the odd and the even lattice site become

$$\begin{aligned}
&(1-\lambda)\psi_{2m} + (1+\lambda)\psi_{2m-2} + C\psi_{2m-1} = 0,\\
&(1+\lambda)\psi_{2m+1} + (1-\lambda)\psi_{2m-1} + C\psi_{2m} = 0,\\
&(1-\lambda)\psi_{2m+2} + (1+\lambda)\psi_{2m} + C\psi_{2m+1} = 0,
\end{aligned} \tag{4}$$

where m is an arbitrary positive integer and $C = E - V\cos(2\pi\beta(2m-1)^{\nu}) = E - V\cos(2\pi\beta(2m)^{\nu}) = E - V\cos(2\pi\beta(2m+1)^{\nu})$ since all the V_i are constant. From Eq. (4) we obtain

$$\psi_{2m+2} + \frac{2+2\lambda^2-C^2}{1-\lambda^2}\psi_{2m} + \psi_{2m-2} = 0. \tag{5}$$

Following the asymptotic heuristic argument [28], we write $\psi_{2m}\sim Z^m$, where Z is a complex quantity. Then Eq. (5) becomes

$$Z^2 + \frac{2+2\lambda^2-C^2}{1-\lambda^2}Z + 1 = 0, \tag{6}$$

to which the complex solutions are

$$Z_{1,2} = \frac{-G\pm\sqrt{G^2-4}}{2} \tag{7}$$

with $G = \frac{2+2\lambda^2-C^2}{1-\lambda^2}$. From Eq. (7) we conclude that the amplitude is complex or extended (due to $|Z|=1$) if $|G|<2$ whereas real or localized if $|G|>2$. Note $C_{\max} = |E|+V$ and $C_{\min} = |E|-V$, so if E and V are fixed, we get

$$\begin{aligned}
G_{\max} &= \frac{2+2\lambda^2-(|E|-V)^2}{1-\lambda^2},\\
G_{\min} &= \frac{2+2\lambda^2-(|E|+V)^2}{1-\lambda^2}.
\end{aligned} \tag{8}$$

The conditions for extended and localized solutions are respectively given by

$$\begin{aligned}
&G_{\max}<2 \Rightarrow 2\lambda+V<|E| (\text{extended}),\\
&G_{\max}>2 \Rightarrow 2\lambda+V>|E| (\text{localized}),\\
&G_{\min}>-2 \Rightarrow 2-V>|E| (\text{extended}),\\
&G_{\min}<-2 \Rightarrow 2-V<|E| (\text{localized}).
\end{aligned} \tag{9}$$

Note that for the existence of the mobility edges there is an implicit condition that $2-V>2\lambda+V$, i.e., $V<1-\lambda$. If this condition is satisfied there will be four mobility edges with $\pm E_{c1} = \pm|2-V|$ and $\pm E_{c2} = \pm|2\lambda+V|$. That is, the model defined by the Hamiltonian (1) with $0<\nu<1$ and $V<1-\lambda$ has localized states at the band center ($-E_{c2}<E<E_{c2}$) and at the band edges ($E>E_{c1}$, $E<-E_{c1}$), extended states at $-E_{c1}<E<-E_{c2}$ and $E_{c2}<E<E_{c1}$.

Eq. (9) also implies that there are four insulator–metal transitions in the system as the energy is increased in magnitude from the band center ($E=0$) to the band edges ($\pm E_{c2}, \pm E_{c1}$).

When $V>1-\lambda$, all the states are localized and there is no extended state, which means that $V=1-\lambda$ is a transition point from a state of mobility edge to a state of all the wave-functions localized.

3. Numerical results

In this section we present detailed numerical analysis to support the theoretical predications given in the previous section, which is exact in the thermodynamic limit. We directly diagonalize the model Hamiltonian (1) to obtain the eigenenergies E and the corresponding wave-functions ψ. After that, it is fairly straightforward to calculate the typical physical quantities used in the disordered system, such as the inverse participation ratio and the Lyapunov exponent, to distinguish the localized and extended states. All the numerical results are in exact agreement with the theoretical predictions in Sec. 2, demonstrating that the asymptotic theory works very well even for the finite-sized systems of which the potential is not strictly a constant.