Physics Letters A 379 (2015) 1681–1688

Contents lists available at ScienceDirect

Physics Letters A

www.elsevier.com/locate/pla

Measurement of a spin-1 system

Antonio Di Lorenzo [a,b,*]

[a] Universidade Federal de Uberlândia, Uberlândia, MG, Brazil
[b] CNR-IMM-UOS Catania (Università), Consiglio Nazionale delle Ricerche, Via Santa Sofia 64, 95123 Catania, Italy

ARTICLE INFO

Article history:
Received 5 November 2014
Received in revised form 9 February 2015
Accepted 24 February 2015
Available online 27 February 2015
Communicated by P.R. Holland

ABSTRACT

We derive exact formulas describing an indirect measurement of a spin-1 system. The results hold for any interaction strength and for an arbitrary output variable $\hat{O}$.

1. Introduction

The simplest non-trivial Hilbert space is the two-dimensional one, which describes a spin 1/2 or a qubit. The measurement of a spin 1/2 as realized in the Stern–Gerlach experiment[1] [1] epitomizes the ideal quantum measurement, even though a realistic description of the measurement involves some complications [3].

The next simplest system in quantum mechanics is provided by a three-dimensional Hilbert space, which can be realized, for instance, by a system with spin one. In the current jargon of quantum information, three-level systems are known as qutrits. They are known to provide higher-security quantum cryptography than qubits [4,5]. Furthermore, it has been demonstrated that qutrits can be efficiently engineered and controlled, by using nonlinear optical techniques on bi-photons [6,7]. Spin-1 systems are important also for fundamental issues, as the Kochen–Specker theorem requires an Hilbert space at least three-dimensional [8]. In this context, the possibility of realizing an arbitrary projective measurement was questioned [9] (we remark that a pure state ψ of a spin-1/2 system is always an eigenstate of a spin component $\mathbf{n}\cdot\mathbf{S}$, so that any projective measurement reduces to the measurement of a spin-component, but the same does not hold for a spin-1 system). This challenge was answered positively [10]. The validity of Kochen–Specker theorem for unsharp measurements on a spin-1 system was also questioned [11–14], and it was shown that the theorem holds if the unsharpness is distributed covariantly [15].

Spin-1 systems are the only ones, besides the spin-1/2 systems, that satisfy a generalized idempotence relation $S^3 = S$. To the best of my knowledge, there is no study of the general (i.e., non-projective) measurement of a spin-1, while a spin-1/2 has been treated quite extensively [3,16–18]. In this manuscript, I am going to fill this gap, by studying a measurement of a spin-1 system followed, possibly, by a post-selection [19]. General measurements, i.e., Positive-Operator Valued measures, are discussed in the books [20–23]; in particular, non-demolition measurements were treated in [24]. Here, we shall consider linear non-demolition measurements, which means that the coupling between the detector and the system is linear in the measured operator $\hat{S}$, and that the latter is conserved during the measurement process.

In principle, for the special case of a detector having a continuous output, one could use the exact formal solution developed by Dressel and Jordan in [25,26], where the final density matrix of the system is expressed in terms of the initial density matrix and the initial Wigner function of the probe. However, these results apply only to the case when the readout variable of the detector is either canonically conjugated to or coincides with the variable appearing in the interaction, and the expression requires expanding the Wigner function in a series of its second argument, and then resumming, if possible, all the terms in the series. While for a spin-1 system it is possible to do so, as we show in Appendix A, the procedure is unnecessarily complicated, and the more straightforward approach used here is better suited to the task. In Appendix B, we provide a slight improvement on the general formula of [25], by showing that it can be expressed in terms of the quantum characteristic function.

Finally, as our results apply to a detector having a discrete spectrum, they may be useful in Nuclear Magnetic Resonance implementations, where two nuclei, one with spin 1 the other with spin $S \geq 1$, interact. A recent realization of weak measurements

* Correspondence to: Universidade Federal de Uberlândia, Uberlândia, MG, Brazil.
E-mail address: dilorenzo@infis.ufu.br.

[1] We remark that the Stern–Gerlach experiment provided the earliest direct evidence for the existence of spin, even though the hypothesis of spin would be advanced by Pauli only in 1924, two years after the experiment. Actually, Stern and Gerlach believed that the silver atoms had an angular momentum $L = 1$, and their goal was to verify Bohr's prediction that the possible values of L_z are quantized. The fact that the line corresponding to $L_z = 0$ was missing from the experiment was overlooked. See [2] for a recent historical account.

http://dx.doi.org/10.1016/j.physleta.2015.02.039
0375-9601/

(which is a limiting case of the results we present below) in NMR was reported in [27].

2. Background

2.1. A useful property of a spin-one operator

In the following, we shall exploit the formula valid for a spin 1,

$$\exp(i\phi\hat{S}) = 1 + i\sin(\phi)\,\hat{S} - [1-\cos(\phi)]\hat{S}^2, \tag{1}$$

which follows from

$$\hat{S}^3 = \hat{S}. \tag{2}$$

We remark that this is the only property of the spin-one operator that we are going to exploit, so that the results presented here apply to any operator satisfying (2), not only operators on qutrits. In other words, the results of the present manuscript apply to any operator having eigenvalues in the set $\{-1, 0, 1\}$. Furthermore, the results can be trivially extended to any operator $\hat{X}$ having three equally spaced eigenvalues x_1, x_2, x_3, $x_2 - x_1 = x_3 - x_2 = \Delta x$, by making the shift and rescaling $\hat{X} = \Delta x\hat{S} + x_2$.

In particular, an operator satisfying $\hat{S}^2 = 1$, e.g. a Pauli matrix representing a spin-1/2, satisfies also (2), so that the following results apply to this case as well, after applying the further restriction $\hat{S}^2 = 1$. As the exact solution of a measurement of a spin-1/2 is well known [16–18,28–30], it will provide a reference check. Furthermore, a projection operator satisfies as well (2), but $\hat{S}^2 = \hat{S}$. Thus, the results presented in the following subsume both those for the measurement a spin-1/2 and those for the measurement of a *yes/no* operator.

Another example of particular relevance where (2) holds is that of two spin 1/2. Their total spin is a 4×4 matrix, giving a reducible representation of SU(2). The sector corresponding to the singlet is represented by the scalar 0, while the sector corresponding to the total spin 1 is represented by a 3×3 operator S_3, namely

$$S = \begin{pmatrix} 0 & \mathbf{0}_3^\dagger \\ \mathbf{0}_3 & S_3 \end{pmatrix} \tag{3}$$

with $\mathbf{0}_3$ the null vector in three dimensions.

Recently, Aharonov et al. have proposed to realize a quantum Cheshire cat [31] by measuring the presence of a particle at a location, and its polarization at a separated location. In this case, in the first location, a yes/no measurement is occurring, while in the second location a measurement of a local spin operator σ is taking place. The latter operator can have the values $+1$ or -1 if the particle is there, and the value 0 if the particle is not there. Therefore, the results presented in the following are relevant to extend the study of the quantum Cheshire cat to an arbitrary coupling [32].

2.2. Description of the measurement

In a measurement, before the interaction, the system and the detector are assumed to be uncorrelated, having a density matrix

$$\rho^- = \rho_i \otimes \rho_{det}; \tag{4}$$

the evolution operator of the system and the detector is taken to be the von Neumann interaction

$$U = \exp(i\hat{Q}\,\hat{S}), \tag{5}$$

with $\hat{Q}$ an operator on the Hilbert space of the detector. The final entangled density matrix is thus

$$\rho^+ = \exp(i\hat{Q}\,\hat{S})\,(\rho_i \otimes \rho_{det})\exp(-i\hat{Q}\,\hat{S}). \tag{6}$$

We shall call the procedure a canonical measurement when the readout $\hat{P}$ has eigenstates $|j\rangle$ such that $\exp(i\hat{Q}\,S)$ translates one of them, say $|j_0\rangle$, into distinct eigenstates $|j_S\rangle$, with S eigenvalues of the measured operator. Furthermore, we shall call the measurement ideal when the detector is prepared initially in the state $\rho_{det} = |j_0\rangle\langle j_0|$. A von Neumann measurement is an ideal canonical measurement. In the present manuscript, however, we shall consider measurements that obey (5), and we shall not make the hypotheses of a canonical and ideal measurement, unless otherwise specified. Thus, we are using a von Neumann interaction, but we are dropping any further hypothesis behind the von Neumann model of measurement. In the case that the observable $\hat{P}$ of the detector is not canonically conjugated to $\hat{Q}$, the procedure could not be properly called a measurement, but perhaps an observation, in the sense that observing $\hat{P}$ reveals something about the system, even though it is not a measurement of any observable $\hat{S}$. In particular, e.g., we could have $\hat{P} = \hat{Q}$, so that the variable does not change with the time-evolution operator $U = \exp(i\hat{Q}\,\hat{S})$. In this case, observing $\hat{Q}$ does not yield information about $\hat{S}$, but about the "logarithmic directional derivative of the post-selection probability along the flow generated by the unitary action of the operator $\hat{S}$" [26].

2.3. Post-selection

The system may be postselected in a state E_f, represented by a positive operator not necessarily having trace one [19,33–36] by making a subsequent measurement. More precisely, the post-selected state is the normalized semipositive definite operator $E_f/\mathrm{Tr}(E_f)$, which allows to make retrodictions about the past behavior of the system and which differs, in general, from the predictive state after the measurement ρ_f. Indeed, the two states coincide only if the post-selection measurement is a projective one. For instance, one could make a projective measurement of an observable $\hat{S}_f$, and analyze the output of the detector separately for each possible outcome S_f [19]. In this case, the post-selection states are the projectors $E_f = |S_f\rangle\langle S_f|$; or one could make a POV measurement of the system [37], then E_f are not necessarily projectors; or, still, one could make a probabilistic post-selection of the data [29].

The reduced density matrix of the detector, for a given post-selection, is

$$\rho_{det|f} = \frac{\mathrm{Tr}_{sys}[(E_f \otimes \mathbb{1})\rho^+]}{\mathrm{Tr}_{sys,det}[(E_f \otimes \mathbb{1})\rho^+]} \tag{7}$$

with Tr the trace, and Tr_{sys} the partial trace on the Hilbert space of the system. The normalization factor $\mathrm{Tr}_{sys,det}[(E_f \otimes \mathbb{1})\rho^+]$ is the probability of successful post-selection $\mathcal{P}_f$.

3. Results

3.1. General formula

Usually, the output to be observed in the detector is $\hat{P}$, the variable conjugated to $\hat{Q}$. This implicitly requires that the detector has an infinite-dimensional Hilbert space, so that one can define canonically conjugated position and momentum operators. We shall not make this assumption and let, instead, the Hilbert space of the detector to be arbitrary.

Let us start by computing the probability of post-selection. After substitution of (1) into (6), and expressing the trace over the detector Hilbert space in terms of position eigenstate, $\mathrm{Tr}_{det}[\ldots] = \int dQ\,\langle Q|\ldots|Q\rangle$, we have

$$\mathcal{P}_f = \omega\left\{1 - 2\bar{\hat{s}}A''_w + \overline{\hat{s}^2}B_w - 2\bar{\hat{t}}C'_w + 2\overline{\hat{s}\hat{t}}D''_w + \overline{\hat{t}^2}E_w\right\}, \tag{8}$$