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An accurate algorithm to calculate the Hurst exponent of self-similar processes

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ABSTRACT

In this paper, we introduce a new approach which generalizes the **GM2** algorithm (introduced in Sánchez-Granero et al. (2008) [52]) as well as fractal dimension algorithms (**FD1**, **FD2** and **FD3**) (first appeared in Sánchez-Granero et al. (2012) [51]), providing an accurate algorithm to calculate the Hurst exponent of self-similar processes. We prove that this algorithm performs properly in the case of short time series when fractional Brownian motions and Lévy stable motions are considered.

We conclude the paper with a dynamic study of the Hurst exponent evolution in the S&P500 index stocks.

1. Introduction

The discussion about the underlying mechanism that governs stock markets remains still open for the last five decades. Thus, in financial literature coexist papers that defend the market efficiency with others that reject the hypothesis of efficiency.

From the beginning, several authors questioned the Efficient Market Hypothesis (EMH) (see [10,24,30]) and meanly specific aspects such as the consequences of using the normal distribution. The normal distribution allows to explain the evolution of stock market prices under the EMH and it has been the basis on which several famous market theories such as Mean Variance Portfolio Selection Theory [36,37], Capital Asset Pricing Theory [53], the Black–Scholes Model for Options Valuation [6,41] or the Modigliani & Miller Capital Structure Theory [42,43] have been developed.

However, the collection of works looking for an alternative market theory has grown since the nineties. In this way, the Fractal Market Hypothesis (FMH) introduced by Peters in [49] is the most popular among researchers due to its robustness (see [5,7,8,32,34,59]). FMH focus on the concept of liquidity which provides smooth prices processes in the market. In illiquid scenarios, market becomes unstable and extreme movements may occur. Furthermore, the FMH is also connected to both ideas of fractality and multifractality in market prices movements. Works about multifractality in stock markets have increased in the last years so we can quote [9,28,29,45] among others.

Therefore, the discussion about market efficiency has become a classical topic in finance. Indeed, it is related in some way with a Brownian motion as a model for the logarithm of the price of a stock (see for instance, [26,27]). On the other hand, an alternative is the study of long-memory in the series, which is usually done by assuming a fractional Brownian motion as a model for the logarithm of the price of a stock. To study long-memory, researchers use the Hurst exponent and explicitly or implicitly, a fractional Brownian motion is used as a model of the series (see for example, [17,18,25,48]). There is even another research line which involves the use of Lévy stable motions as a model for the logarithm of the price of a stock. This model is especially interesting for stocks with strong movements, since it can account for a great volatility (see

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for example, [13,14,22,46]). All the previous models to describe the evolution of financial series are particular cases of self-similar processes (first appeared in [23]). In this way, recall that (fractional) Brownian motions are particular cases of Lévy stable motions. For the definition of these usual self-similar processes, we refer the reader to [15, Section 8.1], or the paper [14, Section 2].

The concept of market memory connects both the EMH and the FMH. Thus, the presence of memory in the market prices implies that the market is not efficient since the random walk hypothesis is automatically rejected.

The two most popular classical methods to explore market price memory are the R/S analysis, based on Hurst's work [20] and introduced in finance by Mandelbrot and Wallis [35], and the DFA introduced in [47]. Nevertheless, these methods have been widely criticized due to its lack of accuracy when the length of the time series is too short which is the case of financial time series (see for instance [26,52,57,58]). Other alternative techniques applied to deal with this problem are the Hudak's Semiparametric Method (GPH) [16], the Quasi Maximum Likelihood analysis (QML) [19], the Generalized Hurst Exponent (GHE) [2], the Periodogram Method [54], wavelets [56], the Centered Moving Average (CMA) [1], the multifractal detrended fluctuation analysis (MF-DFA) [21], non-linear tools such as the Lyapunov exponent [4,11], and recently, geometric method-based procedures [52] and fractal dimension algorithms [51]. Note that some of the previous methods are valid to study long-memory for fractional Brownian motions, others are also valid to study memory for Lévy stable motions, while only some of them work for the more general self-similar processes, as it is the case with the method we introduce in this paper.

The organization of the paper is as follows. In Section 2, we recall the definition and some properties of self-similar processes. In Section 3, we introduce a new approach that allows a common framework for geometric method-based procedures and fractal dimension algorithms, while in Section 4 we make some comments on the implementation of the algorithm. In Section 5, we test the new approach with fractional Brownian motions and Lévy stable motions and show that the algorithm works fine with those motions for any Hurst exponent value. Finally, in Section 6 we show a brief historic study of the Hurst exponent of the stocks in the S&P500 index and in Section 7 we present the main conclusions.

2. Self-similar processes and their increments

The results, definitions and notations that we recall next come from the theories of probability and stochastic processes and are necessary to formalize our ideas from a mathematical point of view. In this paper, we introduce a new accurate algorithm to efficiently estimate the self-similarity exponent of self-similar processes, first introduced in [23]. Recall that this wide class of processes includes the classical Brownian motions as well as some of their generalizations such as Lévy stable motions. In this way, some useful references are [12,14,33].

First of all, let $(\mathbf{X}, \mathcal{A}, P)$ be a probability space and let $t \in [0, \infty)$ denote time. It is said that the collection $\mathbf{X} = \{X(t,\omega) : t \geq 0\}$ is a random process or a random function if $X(t,\omega)$ is a random variable for all $t \geq 0$ and all $\omega \in \Omega$ (ω belongs to a sample space Ω). Hence, we can think of $\mathbf{X}$ as defining a sample function $t \mapsto X(t,\omega)$ for all $\omega \in \Omega$. Accordingly, the points of Ω parametrize the functions $\mathbf{X} : [0,\infty) \to \mathbb{R}$ and P is a probability measure on this class of functions.

Let $X(t,\omega)$ and $Y(t,\omega)$ be two random functions. We write $X(t,\omega) \sim Y(t,\omega)$ to denote that the two preceding random functions have the same finite joint distribution functions.

Additionally, let us also recall the next definition. It provides the description of a wide range of random processes which play a key role in the study of long-memory of financial time series as well as some properties about the increments of that random functions.

Definition 2.1.

(1) (See [23].) A random process $\mathbf{X} = \{X(t,\omega) : t \geq 0\}$ is said to be H-self-similar if for some parameter $H > 0$, it is verified that

$$X(at,\omega) \sim a^H X(t,\omega), \tag{1}$$

for all $a > 0$ and $t \geq 0$. Here, the parameter H is called the self-similarity index or exponent of $\mathbf{X}$. So, Eq. (1) states that every change of time scale $a > 0$ leads to a change of space scale a^H in the case of H-self-similar random processes. Further, the bigger H, the more drastic is the change of the space variable. Note that Eq. (1) provides a scale-invariance of the finite dimensional distribution of $\mathbf{X}$.

(2) (See [33].) The increments of a random function $X(t,\omega)$ are said to be:

(a) stationary, if for all $a > 0$ and all $t \geq 0$, it is verified that

$$X(a+t,\omega) - X(a,\omega) \sim X(t,\omega) - X(0,\omega).$$

(b) self-affine with parameter $H \geq 0$, if for any $h > 0$ and any $t_0 \geq 0$,

$$\begin{aligned} &X(t_0+\tau,\omega) - X(t_0,\omega) \\ &\quad \sim \frac{1}{h^H}\big\{X(t_0+h\tau,\omega) - X(t_0,\omega)\big\}. \end{aligned} \tag{2}$$

If $X(t,\omega) \sim Y(t,\omega)$, then we have that $X(t,\omega) - X(t_0,\omega)$ is a semistable stochastic process in the sense of Lamperti [23] for all $t \geq t_0$ (see [33]). The semistability of a random process is a weaker property than the self-affinity of the corresponding increments, since it is verified that if a random process $\mathbf{X}$ is semistable with parameter H and has stationary increments, then $\mathbf{X}$ is the restriction to $t \geq 0$ of a random process with self-affine increments with parameter H [33].

Moreover, due to [33, Corollary 3.6], if a random function $X(t,\omega)$ has self-affine increments with parameter H, then a T^H-law is verified as follows:

$$M(T,\omega) \sim T^H M(1,\omega), \tag{3}$$

where $M(T,\omega)$ is the T-period cumulative range of the random function $X(t,\omega)$, given by

$$\begin{aligned} M(t,T,\omega) = &\sup_{s\in[t,t+T]}\big\{X(s,\omega) - X(t,\omega)\big\} \\ &- \inf_{s\in[t,t+T]}\big\{X(s,\omega) - X(t,\omega)\big\}, \end{aligned} \tag{4}$$

and $M(T,\omega) = M(0,T,\omega)$. In particular, if $\mathbf{X}$ is a fractional Brownian motion (FBM for short), then we can replace sup and inf by max and min, respectively, in Eq. (4) (see [33, Proposition 4.1]).

Another interesting result we would like to point out establishes that any H-self-similar random process with stationary increments has self-affine increments with parameter H (see [55, Lemma 3.4]), so Eq. (3) is again verified. But what is more: the reciprocal is also true, that is, any random process $\mathbf{X}$ with self-affine increments with parameter H and such that $X(0,\omega) = 0$ is H-self-similar. Indeed, just take $t_0 = 0$ in Definition 2.1 (2)(b).

3. Introducing the FD approach

This section has several purposes. First, we mathematically motivate and introduce our new approach which will allow us to accurately estimate the Hurst exponent of self-similar processes (as