Physics Letters A 378 (2014) 2377–2381

Contents lists available at ScienceDirect

Physics Letters A

www.elsevier.com/locate/pla

Rogue waves: New forms enabled by GPU computing

Christopher Chabalko [a,*], Ayan Moitra [a], Balakumar Balachandran [b]

[a] M0102 Glenn Martin Mezzanine (089), University of Maryland, College Park, MD 20742-5031, United States
[b] 2181 Glenn Martin Hall (088), University of Maryland, College Park, MD 20742-5031, United States

ARTICLE INFO

Article history:
Received 24 January 2014
Received in revised form 6 June 2014
Accepted 7 June 2014
Available online 12 June 2014
Communicated by C.R. Doering

Keywords:
Nonlinear Schrödinger equation
Rogue waves
GPGPU

ABSTRACT

A method to determine parameters governing periodic Riemann theta function rogue-wave solutions to the nonlinear Schrödinger equation is presented. A map of parameter values leading to candidate solutions is developed. In addition to candidate solutions, an overview of qualitative aspects of the solution space can be gained from this map. Based on these findings, several new extreme wave solutions are presented. Although the computations required to determine the map are quite demanding, it is shown that these computations can be efficiently accelerated with a parallel computing architecture. A general purpose computing on a graphics processor unit (GPGPU) implementation yielded a 400× acceleration over a single threaded high level implementation. This acceleration enabled exploration and examination of the solution space, which otherwise would not have been possible. In addition, the solution methodology presented here can be extended to explore other classes of solutions.

1. Introduction

The nonlinear Schrödinger equation (NLSE) has been used to model extreme waves in many domains. The NLSE admits the Benjamin–Feir (modulational) instability [1], which has been proposed as one of the mechanisms for rogue-wave formation [2]. For this reason, it is often used as a model for extreme wave behavior. Several families of analytical solutions have been determined for the NLSE. Much of the motivation in developing these solutions is to model physical rogue waves. These waves can appear on the surface of the ocean, in fiber optical systems, or in other domains as well.

Some of the first analytical solutions to the NLSE were presented by Tracy [3]. Other contributions to determining analytical solutions of the NLSE include those due to Akhmediev and Korneev [4], who determined a family of single parameter solutions. Based on finite gap integration, Smirnov [5] constructed a family of two-gap solutions and derived conditions under which they behave as rogue waves. He extended this work to periodic two phase and three phase solutions in another study [6]. He was able to show that these solutions lead to rogue-wave type behavior when the eigenvalues are close and have large imaginary components. In addition, Smirnov [7] presented rogue wave type elliptic solutions to the NLSE. These solutions, based on three parameters, are distinct from Akhmediev's previously discovered elliptic solutions, which are not considered to be rogue-wave type solutions. Furthermore, the three parameter elliptic rogue-wave solutions degenerate to Akhmediev breathers and Peregrine solitons for certain choices of parameters. A review of nonlinear optical waves, including exact solutions to the nonlinear Schrödinger equation, nonlinear interference, and soliton behavior in dispersive media is available in the book by Akhmediev and Ankiewicz [8]. Different groups have determined other families of rogue-wave type solutions to the standard NLSE. Notably, Akhmediev, Soto-Crespo, and Ankiewicz [9] identify the interference of Akhmediev breathers (ABs) as leading to a type of rogue-wave solution. They show that properly phased AB collisions can result in rogue waves and suggest it as a method to explain and possibly provoke rogue waves in optical fibers.

Akhmediev, Ankiewicz, and Soto-Crespo [10] have also determined a family of rational solutions to the NLSE. The rational solutions are determined by taking a modified Darboux transform of a specially chosen seed solution. They have successfully tested for the presence of rational solutions in a randomly perturbed wave field. A system governed by the NLSE was excited with a plane wave with random perturbations. Regions of large amplitudes were identified and found to match almost identically to the envelope predicted by the rational solutions.

Several groups have verified the analytically predicted solutions with experimental results. The Peregrine soliton, which is a limiting form of several families of analytical rogue-wave solutions, has been studied in a fiber optic cable [11]. In turn, several analytically predicted extreme waves have been demonstrated experimentally in optical fibers [12] and water wave tanks [13,14]. In each case,

* Corresponding author. Tel.: +1 301 405 2377.

http://dx.doi.org/10.1016/j.physleta.2014.06.013
0375-9601/

the observed rogue waves have been modeled after solutions to the NLSE. Finally, Dysthe [15] has introduced a higher order approximation to the wave equation, and this equation is called the Dysthe equation. This equation is considered to provide a more accurate model of extreme wave behavior under certain conditions.

To further contribute to the advancement of the physical realization of rogue-wave solutions to the NLSE, in the present work, the authors offer a method to computationally determine parameters of theta functions which result in as of yet undiscovered rogue waves. While the theta function form of rogue waves is well known [3], the realization of particular theta function solutions enabled by this work enables new physical forms of extreme waves not yet available in the literature. One key aspect of the method, the GPU map, provides an overview of the possible solution parameters. This presentation format provides the analyst with a broad view of the parameter space. Such a broad view can stimulate intuition about governing features of the space, and allow the investigation of sparse regions of the space which would be otherwise difficult to discover.

The scaled NLSE takes the form [16]

$$iu_t - u_{xx} + 2\sigma|u|^2u = 0 \tag{1}$$

where $u(x,t)$ is the complex wave envelope field, t is time, x is the spatial variable, $i = \sqrt{-1}$, the subscripts indicate the corresponding partial derivatives, $\sigma = -1$ yields the focusing case, and $\sigma = 1$ yields the defocusing case.

The inverse scattering transform was used by Shabat and Zakharov [17] to develop analytic solutions of the NLSE. The solution space can be viewed as having a nonlinear Fourier structure, which is comprised of stable and unstable modes. Nonlinear interactions can occur between these modes based on associated eigenvalues [18]. The unstable modes are potential "rogue-wave" solutions. These modes can be expressed in terms of Riemann-theta functions. In this work, the authors provide a procedure to solve for certain unstable "rogue-wave" modes.

Nayfeh and Balachandran [19] showed how externally forced nonlinear systems may exhibit chaotic dynamics and wave behaviors in a variety of physical systems. Given the current state of understanding of solutions of the NLSE, as of yet unknown rogue-wave solutions may be critical to further the understanding of instabilities and extreme behaviors of many systems. Several solutions to the NLSE are already known, and motivate the search for more solutions. The Peregrine soliton is a well-known extreme wave solution [20]. Akhmediev, Ankiewicz, and Taki [21] have detailed exact expressions for rational solutions, which can be used to describe rogue waves. Ma and Ablowitz [22] have provided a solution methodology for obtaining spectral solutions for periodic boundary conditions for both the focusing and defocusing cases. Here, the authors consider solutions to the focusing NLSE for periodic boundary conditions. A new approach is offered to determine parameters that correspond to rogue-wave solutions. Furthermore, the fundamental steps outlined in this parameter selection approach are not restricted to theta function based solutions. Major contributions of this effort are the following: (a) a description of a general procedure to choose likely Riemann theta function solutions to the NLSE and (b) exploration of the parameter space governing possible wave solutions through GPU computing.

The remainder of the article is organized as follows. In the next section, a map of candidate solutions is developed. Next, the predictor–corrector solution procedure is described. Based on the predictor-correct procedure, several rogue-wave solutions are determined and presented. Following that, concluding remarks are collected and presented together at the end.

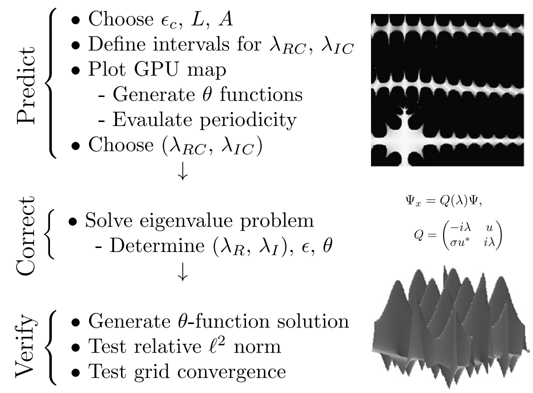

Fig. 1. Flow chart illustrating the procedure to discover Riemann theta function described rogue waves.

2. Rogue-wave formations: computational formulation, solutions, and features

The goal of this work is to present a procedure, based on a predictor–corrector style framework, which allows the discovery of a certain form of rogue-wave solution to the NLSE. The steps involved are summarized in the flowchart given in Fig. 1 and explained with the aid of Eqs. (2) to (18) included in this section. Through GPU computing, this procedure allows an investigator to explore the parameter space, for possible solutions of Eq. (2). In the predictor step, a map of parameters which result in periodic functions, $\psi(x,0)$, is generated. A particular combination of parameters that has a high likelihood of resulting in a rogue wave can then be determined. In the corrector step, the parameters are conclusively refined by solving the spectral eigenvalue problem, shown in Eq. (12), based on the initial guess. In the verification step, a candidate solution is formed by substituting the corrected parameters into Eq. (2). The candidate solution is verified by numerically evaluating the NLSE to determine a residual of zero. The candidate solution is put through two numerical tests, a successful completion of which leads to the acceptance of the candidate solution as a solution. These steps are described in more detail below.

2.1. Predictor

Space periodic spectral solutions to the NLSE can be described by

$$\psi(x,t) = A\frac{\Theta(x,t|\tau,\delta^-)}{\Theta(x,t|\tau,\delta^+)}e^{2iA^2T} \tag{2}$$

where $\Theta(x,t|\tau,\delta^\pm)$ is a Riemann theta function [3,23,24]. A single unstable mode can be considered by taking $\Theta(x,t|\tau,\delta^\pm)$ as a two-dimensional theta function defined as

$$\Theta\left(x,t|\tau,\delta^\pm\right) = \sum_{m_1=-\infty}^{\infty}\sum_{m_2=-\infty}^{\infty}\exp i\left[\sum_{n=1}^{2}m_nK_nx + \sum_{n=1}^{2}m_n\Omega_nT + \sum_{n=1}^{2}m_n\delta_n^\pm + \sum_{j=1}^{2}\sum_{k=1}^{2}m_jm_k\tau_{jk}\right] \tag{3}$$

The parameters governing the theta function (K_n, Ω_n, and $\delta^\pm$) are defined in terms of five spectral parameters A, λ_R, λ_I, ϵ_0, and θ. Following the notation used in earlier work [25], the spectral parameters are defined as

$$\epsilon_1 = \epsilon_0 e^{i\theta}, \qquad \epsilon_2 = \epsilon_1^*, \qquad \sigma_1 = 1, \qquad \sigma_2 = -1 \tag{4}$$

$$\lambda_1 = \lambda_R + i\lambda_I, \qquad \lambda_2 = \lambda_1^* \tag{5}$$