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Impact of the recorded variable on recurrence quantification analysis of flows

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ABSTRACT

Recurrence quantification analysis (RQA) is useful in analyzing dynamical systems from a time series $s(t)$. This paper investigates the robustness of RQA in detecting different dynamical regimes with respect to the recorded variable $s(t)$. RQA was applied to time series $x(t)$, $y(t)$ and $z(t)$ of a drifting Rössler system, which are known to have different observability properties. It was found that some characteristics estimated via RQA are heavily influenced by the choice of $s(t)$ in the case of flows but not in the case of maps.

1. Introduction

Biological systems are complex as they have a huge number of components and degrees of freedom that interact with the environment in subtle ways. Mathematical modeling and analysis of such systems is a challenge. Interactions with the environment or with another system that result in a qualitatively different dynamics, can be represented by a change in a bifurcation parameter. In the case of relatively slow changes, this leads to a drifting behavior which is a common type of nonstationarity, an additional difficulty that has to be dealt with in practice [1–3]. Often, the measurement process itself restricts the type of analyses that can be performed, because it gives access only to a few variables of the system.

The ability of recurrence plots (RPs) and recurrence quantification analysis (RQA) to localize bifurcation behavior in drifting systems, without any a priori hypothesis of the equations of motion, is known [4–7]. By "drifting" we mean slowly time-varying. These features are particularly important in the analysis of data series of biological systems, intrinsically nonstationary. Originally, RPs were introduced to visually distinguish different dynamical behaviors in time series, since periodic, chaotic and random behaviors generate distinct structures in the RPs [4]. Subsequently, RQA was introduced to quantify the properties of RPs [5]. One of the first demonstrations of RQA capabilities was a windowed analysis of the times series of a drifting logistic map, in which several RQA variables were sensitive to bifurcation behavior [6]. The RQA detection of bifurcations, discussed in the context of maps [6], has been applied to time series of flows, not only to detect bifurcations, but also to quantify other dynamical features [3,8,9]. Unfortunately, some characteristics calculated using RQA are very sensitive to user-specified parameters, such as recurrence thresholds, and there is no agreement as how to choose such parameters [7].

Recently, a link was established between observability and embedding theory, demonstrating that the effectiveness of numerical algorithms in quantifying the dynamical features of a system is sometimes highly determined by the variable chosen to reconstruct its dynamics [10–12].

Given the growing importance of RQA in a number of simulated and experimental problems, it is only natural to wonder to what extent does the choice of the recorded variable influence the performance of RQA. In the present work, such robustness is investigated, thus illustrating that observability might influence RQA. In addition to this, another contribution of this work is the investigation of the aforementioned issues in the context of *flows* rather than maps, as discussed in [6]. To this end, the drifting Rössler system is used because of its clear-cut observability properties. The reported results show that, depending on the recorded variable used, some transitions between dynamic regimes are less visible when RQA is performed and some are especially hard to detect.

The present work is organized as follows. Section 2 provides some background. The procedure to generate three time series of the drifting Rössler system, for each of the variables x, y and z is defined in Section 3. Such variables can be ranked as $y \triangleright x \triangleright z$ in terms of observability [11,12]. In Section 4 it is found that some characteristics of RQA are insensitive to bifurcations when the variable z is used. The RQA variables are evaluated using the Poincaré

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section of the flow in Section 5, in order to compare the impact of the variables chosen to reconstruct the dynamics on the RQA of flows and maps. The results are discussed in Section 6 and the main conclusions are pointed out in Section 7.

2. Background

2.1. Observability

In this section we briefly review the observability coefficients as defined in [11]. Consider the autonomous system $\dot{\boldsymbol{x}} = f(\boldsymbol{x})$, where $\boldsymbol{x} \in \mathbb{R}^n$ is the state vector and $f: \mathbb{R}^n \mapsto \mathbb{R}^n$ is the vector field. Consider further the measuring function $h: \mathbb{R}^n \mapsto \mathbb{R}$ such that $s(t) = h(\boldsymbol{x})$, where $s \in \mathbb{R}$ is the observable. Differentiating $s(t)$ yields

$$\dot{s}(t) = \frac{d}{dt}h(\boldsymbol{x}) = \frac{\partial h}{\partial \boldsymbol{x}}\dot{\boldsymbol{x}} = \frac{\partial h}{\partial \boldsymbol{x}}\boldsymbol{f}(\boldsymbol{x}) = \mathcal{L}_f h(\boldsymbol{x}), \tag{1}$$

where $\mathcal{L}_f h(\boldsymbol{x})$ is the Lie derivative of h along the vector field $\boldsymbol{f}$. The jth-order Lie derivative is given by [13, p. 8]:

$$\mathcal{L}_f^j h(\boldsymbol{x}) = \frac{\partial \mathcal{L}_f^{j-1} h(\boldsymbol{x})}{\partial \boldsymbol{x}} \cdot \boldsymbol{f}(\boldsymbol{x}), \tag{2}$$

where $\mathcal{L}_f^0 h(\boldsymbol{x}) = h(\boldsymbol{x})$. The time derivatives of s can be written in terms of Lie derivatives as $s^{(j)} = \mathcal{L}_f^j h(\boldsymbol{x})$. The observability matrix can be written as [14]:

$$\mathcal{O}_s(\boldsymbol{x}) = \begin{bmatrix} \frac{\partial \mathcal{L}_f^0 h(\boldsymbol{x})}{\partial \boldsymbol{x}} \\ \vdots \\ \frac{\partial \mathcal{L}_f^{m-1} h(\boldsymbol{x})}{\partial \boldsymbol{x}} \end{bmatrix}, \tag{3}$$

where the index s indicates that $\mathcal{O}_s(\boldsymbol{x})$ refers to the system observed from $s(t)$. The system is observable from $s(t)$ if $\mathcal{O}_s(\boldsymbol{x})$ is full rank. This classical definition of observability yields "yes–no" answers and poorly observable systems are (correctly) classified as observable.

In order to rank the quality of the system variables in conveying dynamical information, it is helpful to assess how far is $\mathcal{O}_s(\boldsymbol{x})$ from being rank-deficient. This can be achieved computing a coefficient δ_s that quantifies the numerical ill-conditioning of such a matrix along a trajectory $\boldsymbol{x}(t)$ when the recorded variable is $s(t)$. Hence

$$\delta_s(\boldsymbol{x}) = \frac{|\lambda_{\min}[\mathcal{O}_s(\boldsymbol{x})^T\mathcal{O}_s(\boldsymbol{x})]|}{|\lambda_{\max}[\mathcal{O}_s(\boldsymbol{x})^T\mathcal{O}_s(\boldsymbol{x})]|}, \tag{4}$$

where $\lambda_{\max}[\mathcal{O}_s(\boldsymbol{x})^T\mathcal{O}_s(\boldsymbol{x})]$ indicates the maximum eigenvalue of matrix $\mathcal{O}_s(\boldsymbol{x})^T\mathcal{O}_s(\boldsymbol{x})$ estimated at point $\boldsymbol{x}(t)$ (likewise for $\lambda_{\min}$). Then $0 \le \delta(\boldsymbol{x}) \le 1$, and the lower bound is reached when the system is not observable at point $\boldsymbol{x}$. Coefficient $\delta_s(\boldsymbol{x})$ in (4) is a type of condition number of the matrix $\mathcal{O}_s(\boldsymbol{x})^T\mathcal{O}_s(\boldsymbol{x})$. Averaging $\delta_s(\boldsymbol{x})$ along a trajectory over the interval $t \in [0;\ T]$ yields the observability coefficient

$$\delta_s = \frac{1}{T}\sum_{t=0}^{T}\delta_s\big(\boldsymbol{x}(t)\big). \tag{5}$$

The challenges of evaluating observability from data, i.e. without knowing the system equations, have been discussed in [12].

2.2. Recurrence quantification analysis

The RQA technique was initially proposed by Webber and Zbilut [5,15] to quantify the qualitative information of recurrence plots (RP) formulated by Eckmann et al. [4]. Given a time series $\{x\}$, a recurrence matrix $\mathbf{R}_{i,j}(\epsilon)$ is constructed by the N time-ordered embedded vectors in m-dimensional space $\{\vec{X}_i\}_1^N \in \mathbb{R}^m$ by the rule [7]:

$$\mathbf{R}_{i,j}(\epsilon) = \Theta\big(\|\vec{X}_i - \vec{X}_j\| - \epsilon\big), \quad i,j = 1,\ldots,N. \tag{6}$$

A recurrence situation happens when the distance between $\vec{X}_i$ and $\vec{X}_j$ is less than a threshold ϵ. In that case, the Heaviside Θ function returns 1, otherwise it returns 0. The typical RP is a diagram of $\mathbf{R}_{i,j}(\epsilon)$ where black dots are used to indicate the 1s and the 0s are left blank.

The recurrence structure of $\mathbf{R}_{i,j}(\epsilon)$ can be quantified by indices, some of which are presented in the sequel. The density of recurrent points in the RP is the recurrence rate, often expressed in percentage as:

$$\%REC = \frac{1}{N^2}\sum_{i,j=1}^{N}\mathbf{R}_{i,j}(\epsilon) \times 100\%. \tag{7}$$

The so called "determinism" coefficient is the percentage of recurrent points that form diagonal lines with minimum length $l_{\min}$

$$\%DET = \frac{\sum_{l=l_{\min}}^{N} lP(l)}{\sum_{l=1}^{N} lP(l)} \times 100\%, \tag{8}$$

where $P(l)$ is the frequency distribution of diagonal lines of length l parallel to the identity line. $l_{\min}$ must be small because a large value could result in a sparse histogram $P(l)$. On the other hand, $l_{\min}$ should be sufficiently large to exclude the diagonal lines formed by tangential motion of the trajectory in phase space [7].

The Shannon entropy of line segment distributions was defined as

$$ENTR = -\sum_{l=l_{\min}}^{N} p(l)\ln p(l), \tag{9}$$

and is based on the probability that a diagonal line in the RP has length l, that is, $p(l) = P(l)/N_l$, where N_l is the total number of valid lines $(l \ge l_{\min})$.

The inverse of the longest diagonal line is by definition the divergence

$$DIV = \big(\max\big(\{l_i\}_{i=1}^{N_l}\big)\big)^{-1}, \tag{10}$$

which is expected to be correlated to the largest Lyapunov exponent $\lambda_{\max}$ [4,6,16].

An index related to (9) was defined as [17]

$$S = -\sum_{l=1}^{N} P_{\mathrm{nr}}(l)\ln P_{\mathrm{nr}}(l), \tag{11}$$

where $P_{\mathrm{nr}}(l)$ is the number of diagonal lines formed by *nonrecurrent* points $(R_{i,j} = 0)$ divided by the number of *recurrent* points. A correction to the algorithm in [17] is given by the same author at http://www.atomosyd.net/spip.php?article74.

2.3. Numerical setup

In order to investigate the impact of the recorded variable on RQA, a benchmark system plus a set of indices must be chosen.

Trulla et al. [6] applied the RQA to a drifting logistic equation and reported that the indices (7)–(10) are sensitive to the dynamical regime, and therefore can be used to indicate the transitions: periodic–periodic, periodic–chaotic and chaotic–periodic. Hence a drifting system seems adequate as a benchmark for RQA. Because one of the objectives of this work is to investigate the effects