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The inhomogeneous Suslov problem

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ABSTRACT

We consider the Suslov problem of nonholonomic rigid body motion with inhomogeneous constraints. We show that if the direction along which the Suslov constraint is enforced is perpendicular to a principal axis of inertia of the body, then the reduced equations are integrable and, in the generic case, possess a smooth invariant measure. Interestingly, in this generic case, the first integral that permits integration is transcendental and the density of the invariant measure depends on the angular velocities. We also study the Painlevé property of the solutions.

1. Motivation

In this paper we consider the motion of a rigid body under the constraint that a certain component of the angular velocity vector, as seen in the *body* reference frame, is constant. If such a constant vanishes we recover the classical Suslov problem [13]. In other cases, we have an affine or inhomogeneous generalization of the problem.

Our motivation to treat this problem is twofold. On the one hand, it is a toy example for nonholonomic systems with affine constraints. A classical example of such systems is a ball that rolls without slipping on a rotating plane. This type of systems has received attention in the field of nonlinear control theory (see e.g. [11] and references therein).

On the other hand, despite its simplicity, the inhomogeneous Suslov problem is interesting in its own right. Indeed, we prove that, for certain parameter values considered in Section 4.1, the system provides an example of a mechanical problem that possesses an invariant measure whose density depends on the angular velocities, which is an extremely rare phenomenon in mechanics.[1] We also show that the reduced system is integrable due to the existence of a transcendental first integral.

A closely related system to the one considered in this paper is the inhomogeneous Veselova problem. Here one considers the motion of a rigid body subjected to the constraint that a certain component of the angular velocity as seen in the *inertial* reference frame is constant. This system has been considered in [3].

2. Definition of the problem

Consider the motion of a rigid body under its own inertia subjected to the constraint

$$\mathbf{a} \cdot \boldsymbol{\Omega} = K,$$

where $K \in \mathbb{R}$ is constant. In the above, $\mathbf{a} \in \mathbb{R}^3$ is a fixed unit vector in the body frame and $\boldsymbol{\Omega} \in \mathbb{R}^3$ is the angular velocity of the body also written in the body frame. In the case where $K = 0$ we recover the classical nonholonomic Suslov problem.

Apparently Suslov [13] suggested a mechanism to physically implement such a constraint that is described in [1].

Denote by $\mathbb{I}$ the inertia tensor of the body. It is a symmetric, positive definite 3×3 matrix. The equations of motion are obtained via the Lagrange d'Alembert principle that yields

$$\mathbb{I}\dot{\boldsymbol{\Omega}} = \mathbb{I}\boldsymbol{\Omega} \times \boldsymbol{\Omega} + \lambda \mathbf{a}, \tag{1}$$

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[1] For a discussion on the existence of invariant measures for nonholonomic mechanical systems of kinetic type subjected to linear homogeneous constraints, see [5] and the references therein.

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where the Lagrange multiplier λ is determined by the condition that the constraint is satisfied and "$\times$" denotes the vector product in $\mathbb{R}^3$.

Differentiating the constraint and using the equation of motion we obtain

$$\lambda = -\frac{(\mathbb{I}\boldsymbol{\Omega} \times \boldsymbol{\Omega}) \cdot \mathbb{I}^{-1}\mathbf{a}}{\mathbf{a} \cdot \mathbb{I}^{-1}\mathbf{a}}.$$

With the above choice of λ the equations of motion (1) preserve the quantity $\mathbf{a} \cdot \boldsymbol{\Omega}$. The physical system of interest is obtained by considering the motion on the level set $\mathbf{a} \cdot \boldsymbol{\Omega} = K$.

Note that the energy of the system, $H = \frac{1}{2}\mathbb{I}\boldsymbol{\Omega} \cdot \boldsymbol{\Omega}$ is only preserved on the level set $K = 0$. The inhomogeneous constraint adds or takes away energy from the system.

We will assume that the body frame is oriented in such a way that the vector $\mathbf{a} = (0, 0, 1)$. The constraint is then $\Omega_3 = K$. Without loss of generality, we can also assume that the entry I_{12} of the inertia tensor vanishes. Thus, the inertia tensor has the form

$$\mathbb{I} = \begin{pmatrix} I_{11} & 0 & I_{13} \\ 0 & I_{22} & I_{23} \\ I_{13} & I_{23} & I_{33} \end{pmatrix}.$$

In this case we find that the equations for Ω_1, Ω_2 on the level set $\Omega_3 = K$ are given by:

$$\begin{aligned} I_{11}\dot{\Omega}_1 &= -\Omega_2(I_{13}\Omega_1 + I_{23}\Omega_2) + \Omega_2(I_{22}K - I_{33}K) + I_{23}K^2, \\ I_{22}\dot{\Omega}_2 &= \Omega_1(I_{13}\Omega_1 + I_{23}\Omega_2) + \Omega_1(-I_{11}K + I_{33}K) - I_{13}K^2. \end{aligned} \tag{2}$$

The case where $K = 0$ corresponds to the classical Suslov problem that has been studied in detail. In this case there are two distinct cases of qualitative motion.

1. If the vector $\mathbf{a}$ is an eigenvector of the inertia tensor $\mathbb{I}$, then $\mathbb{I}$ is diagonal ($I_{13} = I_{23} = 0$) and the dynamics is trivial. The angular velocity is constant so the body rotates about a fixed axis with constant speed.[2]
2. If the vector $\mathbf{a}$ is not an eigenvector of the inertia tensor $\mathbb{I}$, then the system possesses a straight line of asymptotic equilibria. Using the conservation of energy, the reduced equations of motion are integrated in terms of hyperbolic functions. In this case there is no smooth invariant measure. For a discussion of the motion of the body in this case see [4].

In this note we consider the case where K is non-zero. Note that $\frac{1}{K}$ is a natural time scale for the system, so we introduce the non-dimensional variables

$$\tau = Kt, \qquad \omega_1 = \frac{1}{K}\Omega_1, \qquad \omega_2 = \frac{1}{K}\Omega_2.$$

The system (2) becomes

$$\begin{aligned} I_{11}\omega_1' &= -\omega_2(I_{13}\omega_1 + I_{23}\omega_2) + \omega_2(I_{22} - I_{33}) + I_{23}, \\ I_{22}\omega_2' &= \omega_1(I_{13}\omega_1 + I_{23}\omega_2) + \omega_1(-I_{11} + I_{33}) - I_{13}, \end{aligned} \tag{3}$$

where $' = \frac{d}{d\tau}$.

For the rest of the paper we will analyze the system (3) depending on the position of the vector $\mathbf{a}$ relative to the principal axes of inertia of the body. We consider two cases, the simplest one when the vector $\mathbf{a}$ is an eigenvector of the inertia tensor $\mathbb{I}$, and the second one, when $\mathbf{a}$ belongs to a two-dimensional eigenspace of $\mathbb{I}$ but is not an eigenvector. The analysis for a generic $\mathbf{a}$ will be postponed for a subsequent publication.

[2] We remark that the influence of a constant gravity field can really complicate the dynamics of the body in this case, see e.g. [12] and references therein.

3. Case when a is an eigenvector of $\mathbb{I}$

The simplest case of motion also occurs when $\mathbf{a}$ is an eigenvector of $\mathbb{I}$. In this case $I_{13} = I_{23} = 0$ and the equations of motion (3) become linear:

$$\begin{aligned} I_{11}\omega_1' &= (I_{22} - I_{33})\omega_2, \\ I_{22}\omega_2' &= (-I_{11} + I_{33})\omega_1. \end{aligned}$$

The trace of the associated constant matrix is zero and its determinant equals

$$\frac{(I_{22} - I_{33})(I_{11} - I_{33})}{I_{11}I_{22}}.$$

The above determinant is greater than zero if either I_{11}, $I_{22} > I_{33}$ or I_{11}, $I_{22} < I_{33}$. So we conclude that if $\mathbf{a}$ is an eigenvector of the inertia tensor, along the axis corresponding to the largest or smallest moment of inertia, then we have simple-harmonic motion in the (ω_1, ω_2) plane.

Similarly, if $\mathbf{a}$ points along the axis of middle inertia, then we have a linear saddle in the (ω_1, ω_2) plane. The dynamics in the case where the body has rotational symmetry and some of the principal moments of inertia coincide can be easily understood.

4. Case when a belongs to a two-dimensional eigenspace of $\mathbb{I}$

In this section we consider the case when the vector $\mathbf{a}$ belongs to the two-dimensional space spanned by two of the principal axes of inertia of the body, but is not aligned with any of them. This is equivalent to saying that the vector $\mathbf{a}$ is perpendicular to a principal axis of inertia but without defining one of them.

We suppose that $I_{13} = 0$ but $I_{23} \neq 0$. Under these assumptions, the principal moments of inertia of the body are

$$\begin{aligned} &J_1 = I_{11}, \qquad J_2 = \frac{1}{2}(I_{22} + I_{33}) + \frac{1}{2}\sqrt{(I_{22} - I_{33})^2 + 4I_{23}^2}, \\ &J_3 = \frac{1}{2}(I_{22} + I_{33}) - \frac{1}{2}\sqrt{(I_{22} - I_{33})^2 + 4I_{23}^2}, \end{aligned} \tag{4}$$

and the vector $\mathbf{a}$ belongs to the two-dimensional eigenspace of $\mathbb{I}$ spanned by the principal axes of inertia of the body associated to J_2 and J_3. In other words, $\mathbf{a}$ is orthogonal to the principal axis of inertia associated to J_1.

The equations of motion (3) simplify to:

$$\begin{aligned} J_1\omega_1' &= -I_{23}\omega_2^2 + \omega_2(I_{22} - I_{33}) + I_{23}, \\ I_{22}\omega_2' &= \omega_1\big(I_{23}\omega_2 + (I_{33} - J_1)\big). \end{aligned} \tag{5}$$

The system possesses a particular solution of the form

$$\omega_2 = \frac{J_1 - I_{33}}{I_{23}}, \qquad \omega_1 = -\frac{(J_1 - J_2)(J_1 - J_3)}{I_{23}J_1}t + c_0,$$

where c_0 is an arbitrary constant. Hence, the horizontal line $\omega_2 = \frac{J_1 - I_{33}}{I_{23}}$ is invariant by the flow and so are the semi-planes

$$\omega_2 > \frac{J_1 - I_{33}}{I_{23}} \quad \text{and} \quad \omega_2 < \frac{J_1 - I_{33}}{I_{23}}.$$

At this point, we divide our analysis in two separate cases depending on whether J_1 coincides with either of J_2 or J_3, or not.

4.1. Case when $J_1 \neq J_2, J_3$

Then system (5) possesses the integral of motion