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Effective Kratzer and Coulomb potentials as limit cases of a multiparameter exponential-type potential

J. García-Ravelo [a,*], A. Menéndez [a], J. García-Martínez [a], A. Schulze-Halberg [b]

[a] Departamento de Física, Escuela Superior de Física y Matemáticas, Instituto Politécnico Nacional, Edificio 9, Unidad Profesional Adolfo López Mateos, México D.F., 07738, Mexico

[b] Department of Mathematics and Actuarial Science and Department of Physics, Indiana University Northwest, 3400 Broadway, Gary, IN 46408, USA

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ABSTRACT

We show that the effective Kratzer and Coulomb potentials can be obtained by taking particular limits of a multiparameter exponential potential that was studied recently. Moreover, we demonstrate that the bound state solutions of the exponential potential reduce correctly to their well-known counterparts associated with the Kratzer and Coulomb potentials. As a byproduct, we obtain a new limit relation for the hypergeometric function.

1. Introduction

It is well known that many exactly-solvable Schrödinger equations admit bound state solutions in terms of hypergeometric or confluent hypergeometric functions [1]. Examples for the potentials of such Schrödinger equations are provided by the Coulomb, Kratzer or Morse systems, the bound states of which are expressed through confluent hypergeometric functions. On the other hand, the Hulthén, Scarf, or Eckart potentials admit bound state solutions given by hypergeometric functions. In general, these two groups of potentials are not related, however, the Hulthén potential and the Coulomb potential have been partially linked through their energy spectra [2] and s-states [4], but without discussing [3] the subtle connection between their l-wave functions ($l \neq 0$). The purpose of this article is to show that a limiting process can be used to render both the Kratzer and Coulomb potentials as special cases of a general, multiparameter exponential potential, the bound-state solutions of which are given in terms of hypergeometric functions. More precisely, we consider the singular radial potential of the reference [5], which contains several potentials as particular (non-limiting) forms, one of them being the Hulthén potential. Given the properties of this exponential-type potential, approximate bound-state solutions for the l-waves can be obtained from the corresponding, exact solutions for the s-waves. Using this result, we will prove that the effective Coulomb and Kratzer potentials, together with their bound-state solutions, are limit cases of the exponential potential and its bound-state solutions, respectively.

The remainder of this work is organized as follows. In Section 2, we introduce our multiparameter exponential potential and show that application of a certain parameter limit leads to the well-known Kratzer and Coulomb potentials, as well as to the pseudo-Coulombian potential that was studied in [11]. Section 3 is devoted to the construction of bound-state solutions associated with the limiting cases of the multiparameter exponential potential. To this end, we make use of a new relation between the hypergeometric function and its confluent counterpart, which we prove in Appendix A.

2. Effective Kratzer and Coulomb potentials from an exponential-type potential

We start out by introducing a radial exponential potential V, given by the expression Ref. [5]

$$V(r;..k) = A\frac{e^{-r/k}}{1-e^{-r/k}} + B\frac{e^{-r/k}}{(1-e^{-r/k})^2} + C\frac{e^{-2r/k}}{(1-e^{-r/k})^2}, \qquad (1)$$

where $r \geq 0$ and A, B, C, k are real-valued parameters. The principal idea for extracting the Kratzer and Coulomb potentials from (1)

* Corresponding author. Tel.: +52 55 57296000.
E-mail address: g.ravelo@hotmail.com (J. García-Ravelo).

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is to perform a limit with respect to the parameter k. Let us write (1) as a Taylor series in the variable r/k around zero

$$V(r;..k) = (B+C)\left(\frac{k}{r}\right)^2 + (A-C)\frac{k}{r} + \frac{1}{12}(-6A-B+5C) + O\left(\frac{r}{k}\right). \tag{2}$$

In our approach, it is easy to see that the exponential-type potential (1) becomes another type of potential in the limit $k \to \infty$, for all r. This happens when the coefficients of Eq. (1) are chosen as

$$A = \overline{\alpha}k^{-1}, \qquad B = \overline{\beta}k^{-2} \quad \text{and} \quad C = \overline{\gamma}k^{-2}, \tag{3}$$

where $\overline{\alpha}$, $\overline{\beta}$ and $\overline{\gamma}$ are real-valued constants. By virtue of (2) we have

$$\lim_{k\to\infty} V(r;..k) = \frac{\overline{\beta}+\overline{\gamma}}{r^2} + \frac{\overline{\alpha}}{r} = V(r). \tag{4}$$

This expression has the form of the Kratzer and Coulomb potentials. Observe that (2) is defined on the whole positive real line, despite arising as an approximation of (1) close to zero. This is so, because the new parameters A, B and C depend on k. Then, the Kratzer potential and the Coulomb potential are limit cases of the potential (1), in all the real semi-interval.

Next, we will see that the centrifugal potential term can be interpreted as a parameter limit of an exponential expression. To this end, we verify that the term defined as

$$V_c(r) = \frac{l(l+1)}{r^2}; \quad l = 0, 1, 2, .., l_c, \tag{5}$$

is limit case of a convenient superposition of terms of the potential (1), where l_c is the highest angular momentum for which a bound state exists [6]. In fact, accordingly with [7,8] and [9,10], the following potential

$$V_a(r;..k) = \frac{l(l+1)}{k^2}\left[D_1\frac{e^{-r/k}}{(1-e^{-r/k})^2} + D_2\frac{e^{-2r/k}}{(1-e^{-r/k})^2}\right], \tag{6}$$

where

$$D_1 + D_2 = 1, \tag{7}$$

satisfies

$$\lim_{k\to\infty} V_a(r;..k) = V_c(r). \tag{8}$$

Therefore, the effective potential

$$V_{\text{eff}}(r) = V(r) + V_c(r) = \frac{\overline{\beta}+\overline{\gamma}}{r^2} + \frac{\overline{\alpha}}{r} + \frac{l(l+1)}{r^2}, \tag{9}$$

is obtained from the limit

$$\lim_{k\to\infty} V_l(r;..k) = V_{\text{eff}}(r), \tag{10}$$

where

$$V_l(r;..k) = V(r;..k) + V_a(r;..k). \tag{11}$$

This last potential is characterized by its minimum value

$$V_l(r_0^{(l)};..k) = -\frac{1}{4}\frac{[A+B+D_1k^{-2}l(l+1)]^2}{B+C+k^{-2}l(l+1)}, \tag{12}$$

at

$$r_0^{(l)}(..k) = k\ln\left[1 - 2\frac{B+C+k^{-2}l(l+1)}{A+B+D_1k^{-2}l(l+1)}\right]. \tag{13}$$

Hence, when the assignments (3) are considered, we find that the limit of the potential minimum (12) at (13), matches with the potential minimum of the effective potential (9)

$$\lim_{k\to\infty} V_l(r_0^{(l)};..k) = -\frac{1}{4}\frac{\overline{\alpha}^2}{\overline{\beta}+\overline{\gamma}+l(l+1)}, \tag{14}$$

at

$$\lim_{k\to\infty} r_0^{(l)}(..k) = -\frac{2}{\overline{\alpha}}\left(\overline{\beta}+\overline{\gamma}+l(l+1)\right), \tag{15}$$

when the parameter k tends to infinity.

From Ref. [5], we deduce that the following eigenvalue problem

$$\frac{d^2}{dr^2}\psi_{nl}(r;..k) + \left[E_{nl}(..k) - V_l(r;..k)\right]\psi_{nl}(r;..k) = 0, \tag{16}$$

$$\psi_{nl}(0;..k) = \psi_{nl}(\infty;..k) = 0; \quad n = 0, 1, 2, .. \tag{17}$$

with potential given in Eq. (11), has the eigenvalues

$$E_{nl}(..k) = -\frac{1}{k^2}\left(\frac{(1+n)^2 + (1+2n)\delta_l + k^2(A+B) + D_1l(l+1)}{2(n+\delta_l+1)}\right)^2, \tag{18}$$

with

$$\delta_l = \frac{1}{2}\left[-1 + \sqrt{1 + 4k^2(B+C) + 4l(l+1)}\right]. \tag{19}$$

The corresponding unnormalized eigenfunctions are given by

$$\psi_{nl}(r;..k) = f_0(r;..k)f_1(r;..k)\,{}_2F_1\left(-n, b_{nl}; c_{nl}; e^{-r/k}\right), \tag{20}$$

where

$$f_0(r;..k) = \left(e^{-r/k}\right)^{\frac{c_{nl}-1}{2}}, \qquad f_1(r;..k) = \left(1 - e^{-r/k}\right)^{\delta_l+1}, \tag{21}$$

with

$$b_{nl} = \frac{(n+2\delta_l+1)(\delta_l+1) - k^2(A+B) - D_1l(l+1)}{n+\delta_l+1}, \tag{22}$$

and

$$c_{nl} = -\frac{n^2 + (2\delta_l+1)n + k^2(A+B) + D_1l(l+1)}{n+\delta_l+1} > 1. \tag{23}$$

It is important to remark that when $l = 0$, Eq. (16) represents the radial Schrödinger equation for the potential $V_0\ (r;..k)$ given by Eq. (1). For the case $l \neq 0$, from Eqs. (11) and (8), Eq. (16) can be seen as an approximation to a radial Schrödinger equation with potential (1) plus centrifugal term approximated by the potential $V_a\ (r;..k)$ of Eq. (6). So, as it is indicated in Eq. (10), Eqs. (18) and (20) are a good approximation to the eigensolution of the Schrödinger equation with potential (1) and centrifugal term (5), when the parameter k is large.

Since the function $V_l\ (r;..k)$ is contained in the eigenvalue problem (16)–(17) and satisfies the limit (10), we want to estimate the corresponding limits of the eigenvalues (18) and of the eigenfunctions (20) of the problem. Before continuing we note that by Eq. (3) δ_l in Eq. (19) is independent of k, then the parameters b_{nl}, c_{nl} and the energy spectrum $E_{nl}(..k)$ can be written in short form as

$$b_{nl} = uk + \upsilon, \qquad c_{nl} = uk + w,$$
$$E_{nl}(..k) = -\frac{1}{4k^2}(uk + w - 1)^2, \tag{24}$$