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An exactly soluble model of a shallow double well

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ABSTRACT

Shallow one-dimensional double-well potentials appear in atomic and molecular physics and other fields. Unlike the "deep" wells of macroscopic quantum coherent systems, shallow double wells need not present low-lying two-level systems. We argue that this feature, the absence of a low-lying two-level system in certain shallow double wells, may allow the finding of new test grounds for quantum mechanics in mesoscopic systems. We illustrate the above ideas with a family of shallow double wells obtained from reflectionless potentials through the Darboux–Bäcklund transform.

1. Introduction

In the present Letter we study the properties of an exactly soluble model consisting of a structureless particle on the line subject to a symmetric shallow double-well potential, where by "shallow" we mean that the ground level is above the energy of the classical separatrix ($V(x=0)$). The main purpose of the present Letter is to elucidate in which aspects the behavior of such shallow double wells differs from that of the "deep" double wells that appear, for instance, in the description of Josephson junction based circuits exhibiting macroscopic quantum coherence (MQC) [1–3]. Recently, a symmetric shallow double-well potential has been used in the description of unstable phonon modes of the layered superconductor $LaO_{0.5}F_{0.5}BiS_2$. The model strongly suggests that a dynamical deformation of the structure of the material lies at the heart of its superconductor transition [4]. Previously a similar model, leading to similar conclusions, was proposed for the superconducting composite $MgCNi_3$ [5,6]. The properties of both partially fragmented Bose–Einstein Condensates [7] and high-spin fermionic systems [8] in shallow double-well optical traps are another example of current interest. Symmetric shallow double wells may also play a role in the superionic transition of the AgI compound [9,10], as we shall argue on the following pages.

In order to generate exactly soluble potentials we resorted to the Darboux–Bäcklund Transform [11] (DBT) and the factorization method [12–15]. In a nutshell, a DBT connects two Hamiltonians, self-adjoint second order linear differential operators, say an initial Hamiltonian, H_I with spectrum σ_I, and transformed Hamiltonian, H_T with spectrum σ_T, which can be factorized in terms of a first order differential operator A in the form:

$$H_I = A^\dagger A + \epsilon, \qquad H_T = AA^\dagger + \epsilon, \tag{1}$$

where $\dagger$ stands for the adjoint and ϵ is a real valued constant, the so-called factorization energy. Andrianov and collaborators [15] have shown, in the first term, that neither the ground energy, $E_0^{(I)}$ of H_I, nor that of H_T, $E_0^{(T)}$, is smaller than ϵ, and that there are exactly three different types of DBT, namely: 1) those for which $\epsilon = E_0^{(I)} = E_0^{(T)}$, 2) those for which $\epsilon = E_0^{(I)} < E_0^{(T)}$, and 3) those for which $\epsilon = E_0^{(T)} < E_0^{(I)}$. In the second term, is was shown in the cited article that type (1) transforms are isospectral, *i.e.* that in that case $\sigma_I = \sigma_T$, that type (2) transforms are quasi-isospectral with $\sigma_T = \sigma_I - \{\epsilon\}$ and type (3) transforms are also quasi-isospectral, but with $\sigma_T = \sigma \cup \{\epsilon\}$. This last type of DBT is the only one we shall employ in the present Letter.

As there is an underlying super-symmetric algebra [16] involving H_I, H_T and A, this DBT we have described is preferently

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referred to in the literature as the 1-SUSY (first order super-symmetric) transform. The field of super-symmetric quantum mechanics [15–20], involving not only 1-SUSY but also higher order (n-SUSY) transforms [21–23] (essentially: n times iterated DBT's) has a vast literature of its own, which goes well beyond the scope of the present Letter.

The authors have not been able to find any precedent in the literature where 1-SUSY or n-SUSY have been used specifically for the generation of shallow double wells although there is at least one study of asymmetric double wells, which is explicitly focused on deep wells [24].

It should also be mentioned that double-well models can be constructed without the use of the DBT or the factorization method. In particular, we find interesting that a semi-analytical study of tunneling in double wells can be achieved without having to resort to the effective two-level system model [25].

We have chosen to construct double-well potentials (DWP from now on) starting from reflectionless potentials through a first order quasi-isospectral DBT that generates a single extra level in the spectrum of the transformed potential. In this way it is warranted that the bounded spectra of our DWP's have exactly two levels each, making the analysis of our results particularly simple. As a collateral result the DWP's studied in this Letter are first order super-symmetric (1-SUSY) partners of reflectionless potentials, and thus second order super-symmetric (2-SUSY) partners of the free particle. Then, the examples of DWP's that are about to be discussed belong to the wide family of n-SUSY partners of reflectionless potentials. An extension of this latter family of potentials (with an exotic underlying superalgebra, exhibiting multiple fermionic and multiple bosonic generators) has recently been studied in relation with n-soliton systems [26] and kinks [27,28]. The examples studied in this pages (as well as the totality of n-SUSY partners of reflectionless potentials) are particular cases of this extended family.

The rest of this Letter is structured as follows: In Section 2 we describe the procedure we have used for constructing soluble shallow double wells, and describe some of the features of the resulting potentials. The significance of our results is discussed in Section 2 and finally, some tentative conclusions are advanced in Section 3.

2. Procedure and results

Let us start by considering the dimensionless Hamiltonian

$$\eta = -\frac{d^2}{dx^2} - 2\,\mathrm{sech}^2 x. \tag{2}$$

This $-2\,\mathrm{sech}^2 x$ potential is an example of a reflectionless potential, well known for having one sole bounded level E_0 and a continuous spectrum that starts at $E = 0$ (see, for example, [29]). Operator η can be factorized in the form

$$\eta = A_\epsilon^\dagger A_\epsilon + \epsilon, \tag{3}$$

with the use of the linear first order operators

$$A_\epsilon = -\frac{d}{dx} + \left(\frac{u'_\epsilon}{u_\epsilon}\right)(x) \tag{4}$$

and

$$A_\epsilon^\dagger = \frac{d}{dx} + \left(\frac{u'_\epsilon}{u_\epsilon}\right)(x), \tag{5}$$

taking $\epsilon < 0$ below the ground level, $E'_0 = -1$, of Hamiltonian (2). In (4) and (5), the prime stands for the x derivative, and the so-called *seed function* $u_\epsilon : \mathbb{R} \to \mathbb{R}$, given by

$$u_\epsilon(x) = \sinh\big(\sqrt{|\epsilon|}\,x\big)\tanh(x) - \sqrt{|\epsilon|}\cosh\big(\sqrt{|\epsilon|}\,x\big), \tag{6}$$

is a non-normalizable (thus unphysical) yet node-free solution of the eigenvalue equation

$$\eta u_\epsilon(x) = \epsilon u_\epsilon(x). \tag{7}$$

The new Hamiltonian

$$\Xi_\epsilon = A_\epsilon A_\epsilon^\dagger + \epsilon, \tag{8}$$

which is of the usual Schrödinger form:

$$\Xi_\epsilon = -\frac{d^2}{dx^2} + V_\epsilon(x) \tag{9}$$

with a potential $V_\epsilon : \mathbb{R} \to \mathbb{R}$ that can be readily expressed in terms of u_ϵ as

$$V_\epsilon = 2\left(\frac{u'_\epsilon}{u_\epsilon}\right)^2 - \frac{u''_\epsilon}{u_\epsilon} + \epsilon, \tag{10}$$

has two bounded states, namely, the ground state

$$\psi_0^{(\epsilon)}(x) = \frac{1}{u_\epsilon(x)} \times \left(\int_{-\infty}^{\infty} \frac{dx}{u_\epsilon^2(x)}\right)^{-1/2} \tag{11}$$

with corresponding energy eigenvalue $E_0 = \epsilon$, *i.e.*

$$\Xi_\epsilon \psi_0^{(\epsilon)}(x) = \epsilon \psi_0^{(\epsilon)}(x), \tag{12}$$

and an excited state $\psi_1^{(\epsilon)}(x)$ with corresponding energy eigenvalue $E_1 = E'_0 = -1$, *i.e.*

$$\Xi_\epsilon \psi_1^{(\epsilon)}(x) = -\psi_1^{(\epsilon)}(x). \tag{13}$$

An explicit, yet not particularly illuminating, expression for $\psi_1^{(\epsilon)}(x)$ can be found from the relation

$$\psi_1^{(\epsilon)}(x) \propto A_\epsilon \phi_0(x), \tag{14}$$

where

$$\phi_0(x) = \sqrt{\frac{1}{2}}\,\mathrm{sech}(kx), \tag{15}$$

is the normalized eigensolution of operator η for its ground level $E'_0 = -1$, *i.e.*

$$\eta\phi_0(x) = -\phi_0(x). \tag{16}$$

The correctness of Eq. (12) can be grasped immediately by observing, from definition (6) that operator $A_\epsilon^\dagger$ annihilates $1/u_\epsilon$, *i.e.*

$$A_\epsilon^\dagger\left(\frac{1}{u_\epsilon(x)}\right) = 0, \tag{17}$$

and then plugging $1/u_\epsilon(x)$ at the right of (8). In a similar way, Eq. (16) is proven by inserting (15) in (2). From (15) and (16), Eq. (13) can be proven with the use of intertwining relation

$$\Xi_\epsilon A_\epsilon = A_\epsilon \eta, \tag{18}$$

which in turn is proven by inserting (3) and (8) in (18).

Plugging (5) in (10) gives us, after some algebra,

$$V_\epsilon(x) = \frac{2(1+\epsilon)(-\epsilon + \mathrm{sech}^2 x \sinh^2\sqrt{|\epsilon|}\,x)}{(\tanh x \sinh\sqrt{|\epsilon|}\,x - \sqrt{|\epsilon|}\cosh\sqrt{|\epsilon|}\,x)^2} \tag{19}$$

as an explicit expression for the V_ϵ potentials.

From (19) it is found that relations

$$V_\epsilon(x=0) = 2\epsilon + 2 \tag{20}$$

and

$$\frac{d^2V_\epsilon}{dx^2}(x=0) = 4\big(3 + 4\epsilon + \epsilon^2\big) \tag{21}$$

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