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Obtaining imaginary weak values with a classical apparatus: Applications for the time and frequency domains

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ABSTRACT

Weak measurements with imaginary weak values are reexamined in light of recent experimental results. The shift of the meter, due to the imaginary part of the weak value, is derived via the probability of post-selection, which allows considering the meter as a distribution of a classical variable. The derivation results in a simple relation between the change in the distribution and its variance. By applying this relation to several experimental results, in which the meter involved the time and frequency domains, it is shown to be especially suitable for scenarios of that kind. The practical and conceptual implications of a measurement method, which is based on this relation, are discussed.

1. Introduction

Weak values were introduced, in a seminal paper by Aharonov, Albert and Vaidman [1], as the result of a weak measurement on a pre- and post-selected system. Their ideas were met with some suspicion [2,3] but have been shown to be valid [4]. Since then, they were used for various tasks such as directly measuring quantum states [5–7] or observation of tiny effects [8,9]. Even though their practical benefits were questioned [10], many new schemes for utilizing weak values are being published rather frequently in recent years. A wide range of challenges, such as charge detection [11], measuring small time delays [12–14] or observing Kerr nonlinearity [15], were addressed. Additional improvements, such using orbital-angular momentum [16], were demonstrated and some extensions to the formalism were suggested [17,18].

Recently, weak measurements were demonstrated, using the time and frequency domains, in a number of experiments: improving phase estimation [19,20], measuring velocity [21] and studying atomic spontaneous emission [22]. In all these schemes, imaginary weak values were used in order to make transformations between effects in time and frequency. In [19,20] a time delay was converted to a spectral shift and in [21,22] it was vice versa. The treatment of the time and frequency domains as a measurement device (meter) was originally suggested by Brunner and Simon [12] and it is in some contrast to the usual formalism of weak measurement, where the meter is described using quantum variables such as position and momentum. In case one wishes to treat time and frequencies as quantum variables, some conceptual difficulties might be encountered. In this work, we provide a common theoretical framework for the experimental results, which is focused on the measurement of imaginary weak values. It is based on the use of a classical random variable for describing the meter, rather than a wavefunction. The formalism is general for any scenario involving imaginary weak values, and it can be applied for a wide range of weak measurements schemes.

The weak value of an observable C on a pre- and post-selected system, described by the *two-state vector* $\langle\Phi|\ |\Psi\rangle$, is given by

$$C_w \equiv \frac{\langle\Phi|C|\Psi\rangle}{\langle\Phi|\Psi\rangle}. \tag{1}$$

A few properties of this expression differ it from other values that can be assigned to an observable, like an expectation value or eigenvalues. It can be much larger, if the pre- and post-selection states are nearly orthogonal, and it is complex in general [23]. The imaginary part of the weak value was found to be highly useful for practical goals [24] and its significance was broadly discussed [25]. Imaginary weak values were used in most, if not all, of the experiments showing increased precision.

2. The standard formalism

The standard formalism of weak measurements is based on an interaction between a pre- and post-selected system to a meter, which is also considered as a quantum system. The interaction can be represented using a Hamiltonian

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$$H = g(t)PC, \tag{2}$$

where C is an observable on the system, P is an operator on the meter and $g(t)$ is a coupling function satisfying $\int g(t)dt = k$. If the strength of this interaction is small, the wavefunction of the meter is real valued and the system is pre- and post-selected to $\langle\Phi|\,|\Psi\rangle$, the change in the average of Q, a variable conjugate to P, would be [26]

$$\delta Q = k \,\mathrm{Re}\, C_w, \tag{3}$$

and the change in the average of P would be

$$\delta P = 2k \,\mathrm{Im}\, C_w \,\mathrm{Var}(P), \tag{4}$$

where $\mathrm{Var}(P) = \langle P^2\rangle - \langle P\rangle^2$ is the variance of P. Here, we can consider the average $\langle\bullet\rangle$ to be taken with respect to the initial wavefunction of the meter. Later, we will extend the notion of average to encompass a more statistical distribution.

The shifts (3) and (4) can be derived using the AAV effect, i.e. replacing the operator C in (2) by its weak value and calculating the evolution of the meter under the effective Hamiltonian. In the case C_w is real this Hamiltonian is self-adjoint, which corresponds to a unitary evolution. When C_w is complex, the resulting non unitary evolution of the meter might seem unphysical, especially since P is a constant of motion under the Hamiltonian (2). Below, we will offer an alternative derivation of (4) and show that unlike (3) it does not require interference in the wavefunction of the meter.

3. Derivation of the main result

Let us consider a simpler Hamiltonian

$$H = \tilde{g}(t)C, \tag{5}$$

where $\tilde{g}(t)$ is a coupling function satisfying $\int \tilde{g}(t)dt = \tilde{k}$. With the assignments $\tilde{g}(t) = g(t)P$ and $\tilde{k} = kP$, we can recover the interaction (2), but we can also regard $\tilde{k}$ as a parameter so (5) would operate only on the Hilbert space of the system. Since our interest is in the regime of weak interactions we can assume $\tilde{k} \ll 1$. If the system is initially in a state $|\Psi\rangle$, then after the evolution caused by (5), the probability of finding it in a state $|\Phi\rangle$, for a known $\tilde{k}$ is given by

$$\begin{aligned} P\big(|\Phi\rangle\big|\tilde{k}\big) &= \big|\langle\Phi|e^{-i\tilde{k}C}|\Psi\rangle\big|^2 \\ &= \big|\langle\Phi|\Psi\rangle\big|^2\big(1 + 2\tilde{k}\,\mathrm{Im}\, C_w\big) + O\big(\tilde{k}^2\big). \end{aligned} \tag{6}$$

Now let us consider a situation where the value of $\tilde{k}$ varies according to some distribution $f(\tilde{k})$. This is to say that the experiment is repeated many times and in each run $\tilde{k}$ can obtain a different value, where the probability that $\tilde{k} = x$ is $f(x)$ if $\tilde{k}$ is discrete or $f(x)dx$ if it is continuous. Using this distribution we can calculate different moments of $\tilde{k}$, for example its average is given by $\langle\tilde{k}\rangle = \int \tilde{k} f(\tilde{k})d\tilde{k}$. For the interaction to be weak, $f(\tilde{k})$ should have a significant value only where $|\tilde{k}| \ll 1$ so the average of $\tilde{k}$ for this distribution, or any of its moments, should be small. We can later relax this requirement to have only the width of the distribution small.

A post-selection to $|\Phi\rangle$ means we are interested only in the cases where the system was found in the state $|\Phi\rangle$. Since the probability for this depends on $\tilde{k}$, the post-selection will modify the distribution of $\tilde{k}$. According to Bayes' theorem, the probability to get some value of $\tilde{k}$, given a post-selection $|\Phi\rangle$, is

$$f_\Phi(\tilde{k}) = \frac{f(\tilde{k})P(|\Phi\rangle|\tilde{k})}{P(|\Phi\rangle)} \tag{7}$$

where $P(|\Phi\rangle) = \int P(|\Phi\rangle|\tilde{k})f(\tilde{k})d\tilde{k} \simeq |\langle\Phi|\Psi\rangle|^2(1 + 2\langle\tilde{k}\rangle\,\mathrm{Im}\, C_w)$ is the average probability of post-selection. By inserting (6) into (7) we can calculate the modified average of $\tilde{k}$, up to second order in $\tilde{k}$:

$$\begin{aligned} \langle\tilde{k}\rangle_\Phi &= \int \tilde{k} f_\Phi(\tilde{k})d\tilde{k} \\ &\simeq \frac{\int \tilde{k}(1 + 2\tilde{k}\,\mathrm{Im}\, C_w)f(\tilde{k})d\tilde{k}}{1 + 2\langle\tilde{k}\rangle\,\mathrm{Im}\, C_w} \\ &\simeq \langle\tilde{k}\rangle + 2\,\mathrm{Im}\, C_w\big(\big\langle\tilde{k}^2\big\rangle - \langle\tilde{k}\rangle^2\big). \end{aligned} \tag{8}$$

A quantity of interest for observing some effect in an experiment can be the difference between the post-selected and initial averages

$$\delta\tilde{k} = \langle\tilde{k}\rangle_\Phi - \langle\tilde{k}\rangle \simeq 2\,\mathrm{Im}\, C_w\,\mathrm{Var}(\tilde{k}), \tag{9}$$

where $\mathrm{Var}(\tilde{k}) = \langle\tilde{k}^2\rangle - \langle\tilde{k}\rangle^2 = (\Delta\tilde{k})^2$ is the initial variance of $\tilde{k}$. This simple relation between the change in a parameter and its uncertainty is our main result. It should be noted that this result does not depend on the specific form of $f(\tilde{k})$, i.e. it is not assumed to be, for example, Gaussian. The only assumption, which leads to the absence of higher order terms in the result, is that $f(\tilde{k})$ have significant values only where $\tilde{k}$ is small. This assumption is discussed in details in Section 3.1.

We can see that if $\tilde{k} = kP$, where k is constant and only P varies, the result (9) is the same as (4). The alternative derivation highlights the fact that the variance appearing there is valid for any type of variations, and not only to pure quantum uncertainty. Naturally, quantum mechanics provides a complete description of any system, so one can argue that any variation in the value of a physical quantity is essentially quantum uncertainty. However, considering a fully quantum description can unnecessarily complicate the analysis of an experimental setup. A formalism involving the distribution of a classical parameter can be much simpler than a complete quantum description.

3.1. The regime for the validity of weakness

The result (9) regards only the change and variance of $\tilde{k}$ and thus it is independent of its average. That is to say, if we add some known constant to $\tilde{k}$, the difference between the initial and post selected averages will not be affected, as long as we stay in the regime where $|\tilde{k}| \ll 1$. As we will now show, the result (9) can hold even when $\langle\tilde{k}\rangle$ is not negligible, provided that we take it into account by modifying C_w. By doing this, we can treat separately the known part of $\tilde{k}$, which is its average $\langle\tilde{k}\rangle$, and the unknown part, which is represented by its uncertainty $\Delta\tilde{k}$.

The evolution $U = e^{-i\tilde{k}C}$, caused by (5), can be written as $U = U_1U_2$, where $U_1 = e^{-i(\tilde{k}-\langle\tilde{k}\rangle)C}$ and $U_2 = e^{-i\langle\tilde{k}\rangle C}$. Thus, the probability of post-selection is given by $|\langle\Phi|U_1|\Psi'\rangle|^2$, where $|\Psi'\rangle = U_2|\Psi\rangle$. By repeating the calculations of (6), (7) and (8), we can see that Eq. (9) is unchanged except for the weak value itself, which is given by $C_w \equiv \frac{\langle\Phi|C|\Psi'\rangle}{\langle\Phi|\Psi'\rangle}$. Now, the calculations involved only the deviation $\tilde{k} - \langle\tilde{k}\rangle$ and all moments higher than 2, of this quantity, have been neglected. Each moment was also multiplied by the real or imaginary parts of an expression of the form $((C^n)_w)^m$ for some n, m. Strictly speaking, all these terms have to be small, but in order to see this explicitly, one should specify the distribution $f(\tilde{k})$, the state $\langle\Phi|\ |\Psi\rangle$ and the observable C. However, in case the second moment, $\mathrm{Var}(\tilde{k})$, is large, higher (even) moments cannot be small. Moreover, in order for the weak value expressions to be large, the scalar product in the denominator $\langle\Phi|\Psi'\rangle$, which appears in all of them, have to be small. Thus, a necessary condition for the validity of (9) is that