JID:PLA AID:22598 /SCO Doctopic: Nonlinear science [m5G; v 1.134; Prn:15/05/2014; 10:56] P.1 (1-8)

Physics Letters A ••• (••••) •••–•••

Contents lists available at ScienceDirect

Physics Letters A

www.elsevier.com/locate/pla

Stability of traveling wave solutions to the Whitham equation

Nathan Sanford [a], Keri Kodama [a], John D. Carter [a,*], Henrik Kalisch [b]

[a] Mathematics Department, Seattle University, 901 12th Avenue, Seattle, WA 98122, USA
[b] Department of Mathematics, University of Bergen, Postbox 7800, 5020 Bergen, Norway

ARTICLE INFO

Article history:
Received 21 December 2013
Received in revised form 19 April 2014
Accepted 23 April 2014
Available online xxxx
Communicated by C.R. Doering

Keywords:
Whitham
KdV
Modulational instability
Fourier–Floquet–Hill method
Dispersion
Water waves

ABSTRACT

The Whitham equation was proposed as an alternate model equation for the simplified description of unidirectional wave motion at the surface of an inviscid fluid. An advantage of the Whitham equation over the KdV equation is that it provides a more faithful description of short waves of small amplitude. Recently, Ehrnström and Kalisch [19] established that the Whitham equation admits periodic traveling-wave solutions. The focus of this work is the stability of these solutions. The numerical results presented here suggest that all large-amplitude solutions are unstable, while small-amplitude solutions with large enough wavelength L are stable. Additionally, periodic solutions with wavelength smaller than a certain cut-off period always exhibit modulational instability. The cut-off wavelength is characterized by $kh_0 = 1.145$, where $k = 2\pi/L$ is the wave number and h_0 is the mean fluid depth.

1. Introduction

The water-wave problem concerns the flow of an incompressible inviscid fluid on a horizontal impenetrable bed. The flow is described by the Euler equations and the dynamics of the free surface are of particular interest [27]. There are a number of models which allow the approximate description of the evolution of the free surface without having to provide a complete solution of the fluid flow below the surface. One of the best known of such models is the Korteweg–de Vries (KdV) equation. If the undisturbed depth of the fluid h_0 is taken as the unit of length and the ratio $\sqrt{h_0/g}$ is taken as the unit of time, then the KdV equation is given in non-dimensional form by

$$\eta_t + \eta_x + \frac{3}{2}\eta\eta_x + \frac{1}{6}\eta_{xxx} = 0. \tag{1}$$

It is well known that solutions of this equation provide a fair approximation to the free surface in the long-wave/shallow-water asymptotic limit [26,32]. For waves of amplitude a and wavelength L, this asymptotic limit is characterized by balancing the two small parameters h_0^2/L^2 and a/h_0. Unlike the full water-wave problem, the KdV equation can be solved exactly for a wide range of initial conditions using the inverse scattering transform [1]. One conspicuous difference between the water-wave problem for the full Euler equations and the KdV equation is the velocity of small disturbances of the form $\cos k(x-ct)$ in their respective linearizations about the zero solution. These linear phase speeds are given by

$$c_K = 1 - \frac{1}{6}k^2, \tag{2}$$

for the KdV equation, and

$$(c_E)^2 = \frac{\tanh(k)}{k} \tag{3}$$

for the dimensionless Euler equations. The parameter k is the wavenumber and the wavelength is given by $L = 2\pi/k$.

The linear phase speed in the KdV equation can be obtained from (3) by taking the first two terms in the Taylor expansion around $k = 0$. A comparison of these phase speeds is provided in Fig. 1. It appears immediately that the linear phase speed for the KdV equation approximates the Euler phase speed well for waves of small wavenumber (i.e. long waves), but does a poor job for waves of larger wavenumber (i.e. short waves).

Recognizing this shortcoming of the KdV equation as a water wave model, Whitham [31] proposed an alternative evolution equation featuring the same nonlinearity as the KdV equation, but one branch of the linear phase speed of the Euler equations in the linear part. The equation has the form

* Corresponding author.
E-mail addresses: nathansanford2013@u.northwestern.edu (N. Sanford), kodamak@seattleu.edu (K. Kodama), carterj1@seattleu.edu (J.D. Carter), Henrik.Kalisch@math.uib.no (H. Kalisch).

http://dx.doi.org/10.1016/j.physleta.2014.04.067
0375-9601/

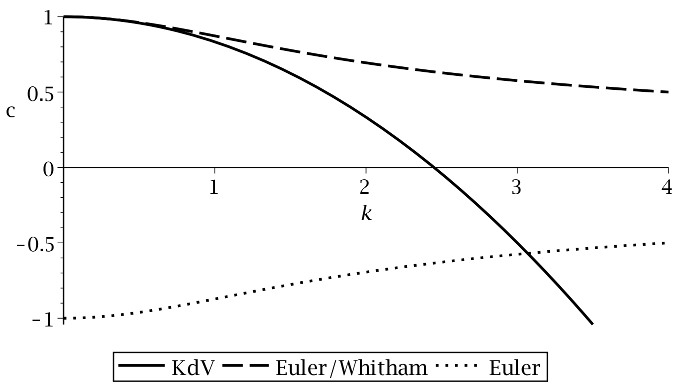

Fig. 1. Phase speed, c, plotted versus wave number, k, for the KdV, Euler, and Whitham equations. The curve for the Whitham phase speed is the same as the curve for the positive part of the Euler phase speed.

$$\eta_t + \frac{3}{2}\eta\eta_x + \frac{1}{2\pi}\int_{-\infty}^{\infty}\sqrt{\tfrac{1}{k}\tanh(k)}\,\hat{\eta}(k,t)e^{ikx}\,\mathrm{d}k = 0. \tag{4}$$

The linear phase speed of the Whitham equation is given by

$$c_W = \sqrt{\frac{\tanh(k)}{k}}. \tag{5}$$

Thus, except for the restriction to one-way propagation, the phase speed of the Whitham equation matches the phase speed of the Euler equations. The linear part of the equation is defined with the help of the Fourier transform,

$$\hat{\eta}(k,t) = \mathcal{F}\{\eta(x,t)\} = \int_{-\infty}^{\infty}\eta(y,t)e^{-iky}\,\mathrm{d}y, \tag{6a}$$

and the inverse Fourier transform,

$$\eta(x,t) = \mathcal{F}^{-1}\{\hat{\eta}(k,t)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{\eta}(k,t)e^{ikx}\,\mathrm{d}k. \tag{6b}$$

In the case where $\eta(\cdot,t)$ is not absolutely integrable, such as if $\eta(\cdot,t)$ is a spatially periodic function, the Fourier transform and the convolution integral in (4) have to be interpreted in the context of tempered distributions [19,29]. It is convenient to define the integral kernel K by

$$\hat{K}(k) = \sqrt{\frac{\tanh(k)}{k}}, \tag{7}$$

so that the Whitham equation can be written in the tidy form

$$\eta_t + \frac{3}{2}\eta\eta_x + K * \eta_x = 0. \tag{8}$$

Even though Eq. (8) has been known for a few decades, it has not been studied as much as the KdV equation. This is partially due to a lack of evidence (beyond the reasoning that led to its derivation) that the Whitham equation actually is a reasonable model for surface water waves. However, there are some recent studies that suggest that the Whitham equation models the evolution of certain waves more accurately than does the KdV equation. In particular, Carter and George [12] study the properties of the Whitham equation as an evolutionary equation and compare its solutions with data from physical experiments, while Borluk et al. [6] investigate the modeling properties in the context of steady waves.

The Whitham equation admits the following conserved quantities

$$\mathcal{Q}_1 = \int_{-\infty}^{\infty}\eta\,\mathrm{d}x, \tag{9a}$$

$$\mathcal{Q}_2 = \int_{-\infty}^{\infty}\eta^2\,\mathrm{d}x, \tag{9b}$$

$$\mathcal{Q}_3 = \int_{-\infty}^{\infty}\left(\eta K * \eta - \eta^3\right)\mathrm{d}x. \tag{9c}$$

These conservation laws are useful in the study of the equation from mathematical point of view [28] and can also be used to test numerical algorithms for the time-dependent problem. Moreover, if the Whitham equation is posed on the real line, then the existence of solitary waves has been recently proven [18] and this result depends strongly on the third conserved quantity (9) which is taken as a mathematical energy in the proof of existence. If periodic solutions are studied, then the domain of integration in the above integrals needs to be replaced by the fundamental periodic domain, such as $[0, L]$ if the solutions are periodic with spatial period L.

In the current work, the focus is on the stability of periodic waves in the Whitham equation and it will be shown numerically that waves of large enough amplitude are always unstable to sideband perturbations. To put this study into context, recall that periodic wavetrains in the full surface water-wave problem may be unstable with respect to modulation by waves of similar but not equal wavelength. In the case of instability, the amplitudes of the so-called sideband modes continue to grow, and a periodic wavetrain literally disintegrates into what seems to be a haphazard combination of waves of various wavelengths. This instability is today known as *modulational instability*, and appears not only in water waves, but also in a range of other dispersive systems. For instance, the instability was first found in nonlinear electromagnetic waves propagating through a liquid [4]. For a historical account of the modulational instability, the reader may consult the recent review by Zakharov and Ostrovsky [33]. In the context of waves on the surface of a body of fluid, Benjamin and Feir [3] established that small but finite-amplitude periodic wavetrains are unstable with respect to a modulational instability. Benjamin [2] found the cut-off separating stable and unstable, small-amplitude wavetrains occurs precisely when the ratio $2\pi h_0/L$ exceeds the value 1.363. For large and intermediate depths, the nonlinear Schrödinger equation and similar models can be used for describing the evolution of wavetrains in the so-called narrow-banded spectrum approximation, and it turns out that the nonlinear Schrödinger equation and most related models feature modulational instability of periodic wavetrains [13,30].

For shallow water waves in the long-wave/shallow-water asymptotic limit mentioned earlier, the generic model equation is the KdV equation, and this equation does not exhibit modulational instability of periodic wavetrains [7]. However, other similar model equations may feature modulational instability, and there have been a number of recent investigations into the modulational stability of periodic solutions of model equations which use mathematical analysis to give definite proofs of stability or instability. See for instance the analyses given in [5,9,24,25]. For a review of asymptotic results on modulational stability, such as the Whitham perturbation method the reader may consult [14].

As explained above, the Whitham equation belongs to a class of equations in which the dispersion relation has been improved (such as the models studied in [21]), but is similar to the KdV equation in the sense that it contains a quadratic nonlinearity. However, since the equation was specifically designed to better approximate short waves, the question arises whether steady