Physics Letters A ••• (••••) •••–•••

Contents lists available at ScienceDirect

Physics Letters A

www.elsevier.com/locate/pla

Integrable motion of curves in self-consistent potentials: Relation to spin systems and soliton equations

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A R T I C L E I N F O

Article history:
Received 8 April 2014
Accepted 12 May 2014
Available online xxxx
Communicated by C.R. Doering

Keywords:
Geometry of moving curves
Integrable spin systems
Soliton equations

A B S T R A C T

Motion of curves and surfaces in $\mathbb{R}^3$ lead to nonlinear evolution equations which are often integrable. They are also intimately connected to the dynamics of spin chains in the continuum limit and integrable soliton systems through geometric and gauge symmetric connections/equivalence. Here we point out the fact that a more general situation in which the curves evolve in the presence of additional self-consistent vector potentials can lead to interesting generalized spin systems with self-consistent potentials or soliton equations with self-consistent potentials. We obtain the general form of the evolution equations of underlying curves and report specific examples of generalized spin chains and soliton equations. These include principal chiral model and various Myrzakulov spin equations in $(1+1)$ dimensions and their geometrically equivalent generalized nonlinear Schrödinger (NLS) family of equations, including Hirota–Maxwell–Bloch equations, all in the presence of self-consistent potential fields. The associated gauge equivalent Lax pairs are also presented to confirm their integrability.

1. Introduction

Integrable soliton equations have interesting geometric connections/equivalence with moving space curves and surfaces both in $(1+1)$ and $(2+1)$ dimensions [1–6]. These connections especially manifest through integrable spin chains. One of the most interesting connections is the mapping of the Heisenberg spin chain onto the integrable nonlinear Schrödinger equation, where the square of curvature of the moving curve is related to the energy density of the spin chain and the torsion is related to the current density [7,8]. Many of the other soliton equations can also be given such connections [1–3]. This relationship can also be reinterpreted as a gauge transformation between the spin systems and soliton equations so that the Lax pairs between the two systems can be mapped onto each other and so also the zero curvature conditions [9,10].

One can also extend this interconnection to moving curves and surfaces in $(2+1)$ dimensions [6,11–14]. In this way one can identify topological conserved quantities with geometrical invariants. For example, one can map Ishimori spin equation and Myrzakulov-I equation with Davey–Stewartson and Zakharov–Strachan $(2+1)$-dimensional nonlinear Schrödinger equations [14], respectively.

In this paper, we present a further generalization by incorporating an additional self-consistent potential in the presence of which the curves and surfaces move. The corresponding generalized evolution equations for the moving curves in $\mathbb{R}^3$ is presented. The generalization to moving surfaces will be presented separately. Several interesting generalized spin systems and soliton equations with self-consistent potentials can then be identified. These include the principal chiral field equation, various generalizations of Myrzakulov family of spin equations [15,16] in $(1+1)$ dimensions and their geometrically equivalent counterparts of generalized nonlinear Schrödinger family of equations in the presence of self-consistent vector fields.

The plan of the paper is as follows. In Section 2, we briefly review the nonlinear dynamics of moving space curves in $(1+1)$ dimensions and deduce the evolution equations for the curvature and torsion of the curve. In Section 3, we generalize the motion equations in the presence of a self-consistent vector potential and deduce the modified form of these equations which now also involve the components of the vector potentials. In Section 4, we identify several specific cases of spin systems in the presence of vector potentials which can be mapped onto the moving curves.

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http://dx.doi.org/10.1016/j.physleta.2014.05.010

These are then transformed into generalized soliton equations. In Section 5, the gauge equivalent Lax pairs are presented for the various systems discussed in the previous section to prove the integrability of them. Then in Section 6 the equivalent induced surfaces are identified. Finally in Section 7, we present our conclusions.

2. Motion of curves in $(1+1)$ dimensions

We consider a space curve in $\mathbb{R}^3$. In $(1+1)$ dimensions the motion of such curves is defined by the following Serret–Frenet equations and rigid body equation [3], respectively,

$$\frac{\partial}{\partial x}\begin{pmatrix}\vec{e}_1\\ \vec{e}_2\\ \vec{e}_3\end{pmatrix}=C\begin{pmatrix}\vec{e}_1\\ \vec{e}_2\\ \vec{e}_3\end{pmatrix},\qquad \frac{\partial}{\partial t}\begin{pmatrix}\vec{e}_1\\ \vec{e}_2\\ \vec{e}_3\end{pmatrix}=G\begin{pmatrix}\vec{e}_1\\ \vec{e}_2\\ \vec{e}_3\end{pmatrix}. \tag{1}$$

Here $\vec{e}_1$, $\vec{e}_2$ and $\vec{e}_3$ are the unit tangent, normal and binormal vectors, respectively, to the curve and x is the arclength parameterizing the curve. The unit tangent vector $\vec{e}_1$ is given by $\vec{e}_1=\frac{\partial\vec{r}}{\partial x}=\frac{1}{\sqrt{g}}\frac{\partial\vec{r}}{\partial\theta}$, where g is the metric $g=\frac{\partial\vec{r}}{\partial\theta}\cdot\frac{\partial\vec{r}}{\partial\theta}$ on the curve such that $x(\theta,t)=\int_0^\theta\sqrt{g(\theta',t)}d\theta'$. Here θ defines a smooth curve and $\vec{r}(\theta,t)$ is the position vector of a point on the curve at time t. In (1) C and G are given by

$$C=\begin{pmatrix}0&\kappa&0\\-\kappa&0&\tau\\0&-\tau&0\end{pmatrix},\qquad G=\begin{pmatrix}0&\omega_3&-\omega_2\\-\omega_3&0&\omega_1\\\omega_2&-\omega_1&0\end{pmatrix}. \tag{2}$$

As it is well known, the curvature and torsion of the curve are given respectively as

$$\kappa=(\vec{e}_{1x}\cdot\vec{e}_{1x})^{\frac{1}{2}},$$
$$\tau=\kappa^{-2}\vec{e}_1\cdot(\vec{e}_{1x}\wedge\vec{e}_{1xx}). \tag{3}$$

The compatibility condition of Eqs. (1) is

$$C_t-G_x+[C,G]=0, \tag{4}$$

or in terms of elements it reads

$$\kappa_t=\omega_{3x}+\tau\omega_2, \tag{5}$$
$$\tau_t=\omega_{1x}-\kappa\omega_2, \tag{6}$$
$$\omega_{2x}=\tau\omega_3-\kappa\omega_1. \tag{7}$$

The above formalism allows one to construct the so-called L-equivalents (Lakshmanan equivalence) of spin systems. Here we present an example which shows how this formalism works. Considering the Heisenberg ferromagnet equation (HFE) [7],

$$\vec{S}_t+\vec{S}\wedge\vec{S}_{xx}=0, \tag{8}$$

where $\vec{S}=(S_1,S_2,S_3)$ is a unit spin vector so that $S_1^2+S_2^2+S_3^2=1$, we assume the identification

$$\vec{e}_1\equiv\vec{S}. \tag{9}$$

Then from the HFE (8) it follows that

$$\omega_1=-\frac{\kappa_{xx}}{\kappa}+\tau^2, \tag{10}$$
$$\omega_2=\kappa_x, \tag{11}$$
$$\omega_3=\kappa\tau. \tag{12}$$

Substituting these expressions of ω_j into Eqs. (5)–(7), we arrive at the system

$$\kappa_t-(\kappa\tau)_x-\kappa_x\tau=0, \tag{13}$$
$$\tau_t+\left(\frac{\kappa_{xx}}{\kappa}\right)_x-2\tau\tau_x+\kappa\kappa_x=0. \tag{14}$$

Let us introduce the following complex function $q=\frac{\kappa}{2}e^{-i\partial_x^{-1}\tau}$. It is easy to check that this function satisfies the well known nonlinear Schrödinger equation [7]

$$iq_t+q_{xx}+2|q|^2q=0. \tag{15}$$

Thus the HFE is L-equivalent to the NSE and vice versa. Our aim in this paper is to construct L-equivalent counterparts of some integrable spin systems with self-consistent potentials in $(1+1)$ dimensions.

3. Moving curves in $(1+1)$ dimensions in the presence of self-consistent potentials

We now introduce a self-consistent vector potential $\vec{W}(x,t)$ in $\mathbb{R}^3$ derivable as

$$\frac{\partial\vec{W}}{\partial x}=\vec{W}_x=2a\vec{W}\times\vec{e}_1, \tag{16}$$

where $\vec{e}_1$ is the unit tangent vector, see Section 2, and a is a constant parameter.

Expressing $\vec{W}$ in the basis of the unit orthonormal triad specifying the moving curve as

$$\vec{W}=W_1(x,t)\vec{e}_1+W_2(x,t)\vec{e}_2+W_3(x,t)\vec{e}_3, \tag{17}$$

the defining equations for the components of the vector potential can be rewritten, after using (1) and (2), as

$$W_{1x}=\kappa W_2, \tag{18}$$
$$W_{2x}=-\kappa W_1+\tau W_3+2aW_3, \tag{19}$$
$$W_{3x}=-\tau W_2-2aW_2. \tag{20}$$

Note that the above three equations imply $\vec{W}^2=W_1^2+W_2^2+W_3^2=C(t)$, where $C(t)$ is a function of t only.

In the presence of the potential field, the evolution equation for the moving curve gets modified due to the self-consistent interaction. The underlying evolution equations for the triad can be identified as follows.

The evolution for the unit tangent vector can be modified in a self-consistent way as

$$\vec{e}_{1t}=\vec{\Omega}\times\vec{e}_1+2a^{-1}\vec{W}\times\vec{e}_1 \tag{21}$$

where (from (1) and (2))

$$\vec{\Omega}=\sum_{i=1}^3\omega_i\vec{e}_i,\qquad \vec{W}=\sum_{i=1}^3 W_i\vec{e}_i. \tag{22}$$

Now using the Serret–Frenet equations for the spatial variation of the trihedral along the arc length, see Eq. (1), one can obtain

$$\vec{e}_{2t}=\vec{\Omega}\times\vec{e}_2+2a^{-1}\left(-W_3\vec{e}_1+\left(W_1-2a\frac{W_3}{\kappa}\right)\vec{e}_3\right) \tag{23}$$

and

$$\vec{e}_{3t}=\vec{\Omega}\times\vec{e}_3+2a^{-1}\left(W_2\vec{e}_1-\left(W_1-2a\frac{W_3}{\kappa}\right)\vec{e}_2\right). \tag{24}$$

In other words, the dynamical equations specifying the unit trihedral gets modified from (1) in the presence of the self-consistent potential $\vec{W}(x,t)$ as