Physics Letters A ••• (••••) •••–•••

Contents lists available at ScienceDirect

Physics Letters A

www.elsevier.com/locate/pla

Quantum state discrimination and enhancement by noise

François Chapeau-Blondeau

Laboratoire Angevin de Recherche en Ingénierie des Systèmes (LARIS), Université d'Angers, 62 avenue Notre Dame du Lac, 49000 Angers, France

ARTICLE INFO

Article history:
Received 12 February 2014
Received in revised form 15 May 2014
Accepted 17 May 2014
Available online xxxx
Communicated by C.R. Doering

Keywords:
Quantum state discrimination
Quantum noise
Quantum detection
Signal detection
Enhancement by noise
Stochastic resonance

ABSTRACT

Discrimination between two quantum states is addressed as a quantum detection process where a measurement with two outcomes is performed and a conclusive binary decision results about the state. The performance is assessed by the overall probability of decision error. Based on the theory of quantum detection, the optimal measurement and its performance are exhibited in general conditions. An application is realized on the qubit, for which generic models of quantum noise can be investigated for their impact on state discrimination from a noisy qubit. The quantum noise acts through random application of Pauli operators on the qubit prior to its measurement. For discrimination from a noisy qubit, various situations are exhibited where reinforcement of the action of the quantum noise can be associated with enhanced performance. Such implications of the quantum noise are analyzed and interpreted in relation to stochastic resonance and enhancement by noise in information processing.

1. Introduction

Quantum states naturally arise when one wants to process, store or retrieve information at the level of quantum objects, such as individual photons, electrons, ions or atoms. Information processing with such quantum systems is a field of recent development, and is currently the subject of intense research, with rich potentialities [1,2]. The statistical theory of information after Shannon has been applied to quantum systems to explore some of their capabilities for information processing and communication [3,4,1,2].

Another direction of recent interest at the interface between physics and information processing is the field of stochastic resonance or effects of enhancement by noise. In its early forms, stochastic resonance represents an enhancement of the response of a dynamical system occurring at an intermediate level of noise [5–7]. Stochastic resonance has progressively been shown feasible in a large variety of forms, in many systems and processes, with various measures of performance receiving enhancement by noise [6,7], and new extensions are regularly reported [8–10]. As a result, in an extended sense we adopt here, stochastic resonance can be understood as a situation where enhancement of the performance in some definite task can be gained from the action of noise. For information processing, stochastic resonance as an enhancement by noise has been reported in different specific tasks, such as signal transmission [11–13], detection [14–19], estimation [20,21], sensor arrays [22–25], or in relation to the statistical theory of information [26–30], although mostly in a classical context. By contrast, stochastic resonance in a quantum context has been addressed by relatively much fewer studies. Early studies on quantum stochastic resonance have considered dynamics in a double-well potential of a time-dependent position operator driven by a periodic forcing and coupled to a heat bath [31–34]. More recently, stochastic resonance has been considered in relation to binary information transmission over noisy quantum channels [35–39]. The analyses of [35–39] exhibit some possibilities of stochastic resonance or enhancement by noise in qubit communication over quantum channels assessed by mutual information, fidelity or transmission rate.

In the present study we will consider an even more basic and fundamental informational operation on quantum systems. We will analyze a discrimination process between two alternative quantum states, which can also be referred to as a quantum detection process [40–48]. In such a binary discrimination or detection process, a quantum system can be in one of two possible states; and from a measurement with two outcomes, a binary decision is taken about which quantum state the system is in [40,41,49]. Concerning quantum state discrimination in general, another distinct problem consists in unambiguous state discrimination [50–52,48]. Unambiguous state discrimination admits measurements that are not conclusive about which state the system is in. By contrast, the type of quantum state discrimination we investigate here, requires a decision about which state the system is in, each time a measurement is performed. In this way, quantum state detection here will designate a conclusive discrimination between two alternative

E-mail address: chapeau@univ-angers.fr.

http://dx.doi.org/10.1016/j.physleta.2014.05.025

quantum states. Imposing a conclusive discrimination exposes to detection errors, and the overall probability of detection error is taken as the performance to be optimized. Based on the theory of quantum detection [40,41], the optimal measurement and its performance are exhibited in general conditions. An application is realized to optimal state discrimination on a qubit. The qubit is a fundamental quantum system of reference with important significance for quantum information. The case of the qubit, which can be worked out in detail, will allow us to test generic models of quantum noise which can affect the discrimination and its performance. The quantum noise is modeled as a noise channel acting on the qubit prior to measurement. Quantum state discrimination in this way is performed from a noisy qubit. The probability of error of the optimal detector operating on the noisy qubit will be analyzed in relation to stochastic resonance and enhancement by noise in information processing. Stochastic resonance is understood here in the broad sense of a noise-enhanced performance, much as for instance in [35–37,39] for the quantum context. Yet the present study here represents the first exploration of its kind of stochastic resonance or favorable noise effects in quantum state detection. Quantum state detection as understood here with no inconclusive measurement, matches the problem of signal detection in the sense of classical (non-quantum) statistical information processing [15,16,49]. Stochastic resonance or enhancement by noise has been shown feasible in classical detection problems [14–19]; and it is investigated here for the first time for quantum detection.

2. Optimal discrimination between two quantum states

As a standard detection situation [41,49], we assume that a quantum system, with complex Hilbert space $\mathcal{H}_N$ of dimension N, can be in one of two possible quantum states. These two quantum states can be pure states or mixed states, and are generally represented by the two (Hermitian positive unit-trace) density operators ρ_0 and ρ_1. The system can be in state ρ_0 or ρ_1 respectively with known prior probabilities P_0 or $P_1 = 1 - P_0$, as a result of its preparation. The detection problem is to determine, from a single non-repeated measurement, whether the quantum system is in state ρ_0 or ρ_1 [41]. A generalized measurement [1] is performed on the system by means of a positive operator-valued measure (POVM) with two elements $\{\mathrm{M}_0, \mathrm{M}_1\}$. Each of the two POVM elements M_k, for $k = 0, 1$, is a positive Hermitian operator satisfying $0 \le \mathrm{M}_k \le \mathbb{1}$, and together summing to the identity operator $\mathbb{1} = \mathrm{M}_0 + \mathrm{M}_1$. When the measurement outcome corresponding to M_k is obtained, then it is decided that the quantum system is in state ρ_k, for $k = 0, 1$. The POVM contains exactly two elements because the detection problem imposes that each time a measurement is performed, a conclusive decision has to be obtained on whether the quantum system is in state ρ_0 or ρ_1. By contrast, unambiguous state discrimination as evoked in the Introduction, would generally include a third POVM element corresponding to the situation where no conclusive decision is returned on the state of the quantum system. Imposing a conclusive measurement usually exposes to detection errors. Except in the special case where the supports of ρ_0 and ρ_1 span orthogonal subspaces, the two quantum states in general cannot be perfectly distinguished, and any conclusive measurement for detection has to cope with some level of error. A relevant task is then to devise optimal strategies to minimize such errors.

The optimal detection strategies under various criteria are characterized in [40,41,43,44]. In this section, we exploit and adapt the results of [40,41] for detection under minimum probability of detection error. Several aspects of such quantum detection have been developed in different directions, for instance in [42,45,47,44,53,46,54]. Here, we consider minimum probability-of-error detection in relation to stochastic resonance and enhancement by quantum noise, which is an original perspective. To obtain the overall probability of detection error, we have the conditional probability of each detection decision which is given by the operator trace [1]

$$\Pr\{\mathrm{M}_k|\rho_j\} = \mathrm{tr}(\rho_j \mathrm{M}_k), \quad j = 0, 1, \; k = 0, 1. \tag{1}$$

The overall probability of detection error $P_{\mathrm{er}} = \Pr\{\mathrm{M}_1|\rho_0\}P_0 + \Pr\{\mathrm{M}_0|\rho_1\}P_1$ then results as

$$P_{\mathrm{er}} = \mathrm{tr}(\rho_0 \mathrm{M}_1)P_0 + \mathrm{tr}(\rho_1 \mathrm{M}_0)P_1 \tag{2}$$

$$= \mathrm{tr}\big[\rho_0 \mathrm{M}_1 P_0 + \rho_1(\mathbb{1} - \mathrm{M}_1)P_1\big] \tag{3}$$

$$= P_1 - \mathrm{tr}\big[(P_1\rho_1 - P_0\rho_0)\mathrm{M}_1\big] \tag{4}$$

since $\mathrm{M}_0 = \mathbb{1} - \mathrm{M}_1$ and the ρ_j's are with unit trace. From Eq. (4), the probability of detection error can also be expressed as

$$P_{\mathrm{er}} = P_1 - \mathrm{tr}(\mathrm{T}\mathrm{M}_1), \tag{5}$$

with the test operator

$$\mathrm{T} = P_1\rho_1 - P_0\rho_0, \tag{6}$$

which is Hermitian but not generally a density operator since T is not positive in general.

We then seek the optimal POVM $\{\mathrm{M}_0 = \mathbb{1} - \mathrm{M}_1, \mathrm{M}_1\}$ that minimizes the probability of detection error P_{er} from Eq. (5). This is achieved by finding the POVM element M_1 that maximizes the term $\mathrm{tr}(\mathrm{T}\mathrm{M}_1)$ in the right-hand side of Eq. (5). To characterize this optimal POVM element, the spectral decomposition of the test operator T is introduced as

$$\mathrm{T} = \sum_{n=1}^{N} \lambda_n |\lambda_n\rangle\langle\lambda_n|, \tag{7}$$

with the eigenvectors $\{|\lambda_n\rangle\}$ of the Hermitian operator T forming an orthonormal basis. One then gets

$$\mathrm{tr}(\mathrm{T}\mathrm{M}_1) = \sum_{n=1}^{N} \lambda_n \,\mathrm{tr}\big(|\lambda_n\rangle\langle\lambda_n|\mathrm{M}_1\big) = \sum_{n=1}^{N} \lambda_n \langle\lambda_n|\mathrm{M}_1|\lambda_n\rangle. \tag{8}$$

Each scalar $\langle\lambda_n|\mathrm{M}_1|\lambda_n\rangle$ in Eq. (8) is a real confined between 0 and 1, since M_1 is a positive operator verifying $0 \le \mathrm{M}_1 \le \mathbb{1}$. For each n, the maximum value of 1 is reached by $\langle\lambda_n|\mathrm{M}_1|\lambda_n\rangle$ when M_1 is the projector $|\lambda_n\rangle\langle\lambda_n|$ on the eigensubspace spanned by $|\lambda_n\rangle$. The POVM element M_1 maximizing the sum in the right-hand side of Eq. (8) is thus realized by summing the rank-one projectors $|\lambda_n\rangle\langle\lambda_n|$ for all the eigenvectors $|\lambda_n\rangle$ associated with a positive eigenvalue λ_n, i.e.

$$\mathrm{M}_1^{\mathrm{opt}} = \sum_{\lambda_n > 0} |\lambda_n\rangle\langle\lambda_n|, \tag{9}$$

achieving for $\mathrm{tr}(\mathrm{T}\mathrm{M}_1)$ in Eq. (8) the maximum $\sum_{\lambda_n>0} \lambda_n$. The optimal POVM element $\mathrm{M}_1^{\mathrm{opt}}$ defined by Eq. (9) is thus the projector on the subspace spanned by the eigenvectors $|\lambda_n\rangle$ associated with all the positive eigenvalues λ_n of the test operator T. The complementary element $\mathrm{M}_0^{\mathrm{opt}} = \mathbb{1} - \mathrm{M}_1^{\mathrm{opt}}$ is the projector on the orthogonal subspace.

The optimal POVM defined by Eq. (9) achieves in Eq. (5) the minimal probability of error

$$P_{\mathrm{er}}^{\mathrm{min}} = P_1 - \sum_{\lambda_n > 0} \lambda_n, \tag{10}$$

which subtracts from the prior P_1 all the positive eigenvalues of the test operator T. Since the eigenvalues λ_n sum to $\mathrm{tr}(\mathrm{T}) = P_1 - P_0$, one has equivalently $P_{\mathrm{er}}^{\mathrm{min}} = P_0 + \sum_{\lambda_n<0} \lambda_n$ summing over