Physics Letters A 378 (2014) 1085–1090

Contents lists available at ScienceDirect

Physics Letters A

www.elsevier.com/locate/pla

Formations of n-order two-soliton bound states in Bose–Einstein condensates with spatiotemporally modulated nonlinearities

Jun-Rong He, Lin Yi *

Department of Physics, Huazhong University of Science and Technology, Wuhan, Hubei 430074, China

ARTICLE INFO

Article history:
Received 28 November 2013
Received in revised form 19 January 2014
Accepted 28 January 2014
Available online 18 February 2014
Communicated by A.R. Bishop

Keywords:
Two-soliton bound state
Harmonic potential
Bose–Einstein condensate

ABSTRACT

The formations of n-order two-soliton bound states (TSBSs) in the Bose–Einstein condensates with spatiotemporally modulated nonlinearities are studied. Exact analytical expressions of the n-order TSBSs are derived by means of the similarity transformations. Further, the numerical simulations are carried out, consistent with the analytical results very well. The stability analysis shows that the solutions can be stable. Our results indicate that the attractive spatiotemporal inhomogeneous nonlinearities can support n-order TSBSs, which has the potential applications to the generation of matter-wave bright solitons in harmonic traps.

1. Introduction

The nonlinear Schrödinger equation (NLSE) has been studied in a diversity of situations. It appears in many branches of physics, including nonlinear optics [1–3], Bose–Einstein condensates (BECs) [4,5], plasma physics [6], hydrodynamics [7], and some organic materials [8]. When the NLSE describes the dynamics of a BEC at zero temperature by the mean-field theory, it often calls the Gross–Pitaevskii equation (GPE) that elucidates the behavior of condensate's macroscopic wave-function [5,9]. Various types of solutions to NLSE and GPE are found of great interest, such as bright (dark) solitons [10–13], periodic traveling waves [14], and localized nonlinear waves [15].

In the last few years, many different types of nonlinear waves have been studied theoretically, experimentally or numerically in the context of BECs [16,17]. The study of inhomogeneous nonlinearities has led to the prediction of many remarkable nonlinear phenomena either for time-dependent [18,19] or space-dependent nonlinear coefficients [20]. Later researches have shown that the spatiotemporal-dependent [15,21] nonlinearities can support explicitly exact solutions of the NLSE. In these situations, the nonlinearity can be controlled by means of the Feshbach-resonance (FR) technique [22] through the s-wave scattering length in BECs and can be achieved by the nonuniform distribution in nonlinear optics [23]. Another particularly important scenario for solitons concerns the NLSE in the case of one dimension with a harmonic potential. This potential is motivated by the purpose of trapping the system in a finite region in space, and/or some time periodically oscillating pattern related to the trapping frequencies in BECs. In this situation, the NLSE has recently gained further importance, mainly because of its direct application to the study of BECs [5,24,25] in fibers and in photonic crystals and other periodic systems [1,19]. There are many methods that have been applied to solving the above-mentioned physical models. It has been shown that the similarity transformations [15,26] are the most useful and interesting ways for dealing with the NLSE with varying coefficients.

On the other hand, as a type of complex soliton structure, the soliton bound states (BSs) have attracted the theoretical and experimental attention. In BEC applications, the soliton is monitored by projecting the BS of $\sim 10^3$ atoms into expulsive harmonic potential; then the soliton was observed propagating without changing of the form for distances of order $\sim$1 mm [11]. In addition, it is shown that during atomic collisions, the atoms can stick together and form BSs in the form of molecules, which is important to the occurrence of an FR [27]. By exploiting an FR to widely tune the interactions between trapped ultracold lithium atoms, BSs can also associate with Efimov trimers [28]. Moreover, the BSs can describe BEC Josephson junctions in optical lattices governed by the GPE [29]. Beyond mean-field description of a dilute gas of bosonic atoms, the BSs may be formed between the coherent condensate and vapor components in the case of attraction between atoms of coupled Hartree–Fock–Bogoliubov equations [30]. In nonlinear optics, BSs (for the first time predicted in [31] in laser models) are observed experimentally in a mode-locked fiber laser, and some of them are characterized as being stable [32,33]. Numerical and experimental studies have confirmed that soliton BSs can exist in the

* Corresponding author.
E-mail address: yilinhust@163.com (L. Yi).

http://dx.doi.org/10.1016/j.physleta.2014.01.050
0375-9601/

amplified-damped fiber systems [34], isotropic Kerr materials [35] and strongly birefringent fibers [36].

The recent papers suggest that BSs of an arbitrary number of solitons can be supported when the harmonic potential is trapped in a designed inhomogeneous nonlinearity [2,3,37,38]. A more specific result is that the two-soliton bound state (TSBSs) can exist in BECs with time-modulated nonlinearity for harmonic potential, both in time-independent and time-dependent trapping frequencies [39,40]. Recently the TSBSs has also been found in a generalized nonautonomous cubic–quintic NLSE with distributed coefficients and time-dependent confining harmonic potentials [41]. Similar structures in optical fibers with a variable dispersion have been found in Ref. [2]. Motivated by the above experimental and theoretical investigations of BSs, in this paper we study the formations of n-order TSBSs in harmonic traps with spatiotemporally modulated nonlinearities. To do so, we make use of the similarity transformations that connect problems with spatiotemporally modulated nonlinearities with simpler ones that have a homogeneous nonlinearity. It is interesting that one obtains the *exact analytical expressions of TSBSs which has infinite numbers*. We show that the attractive spatiotemporal inhomogeneous nonlinearities can support n-order TSBSs in harmonic traps. Numerical simulations are calculated to verify the analytical results. In addition, we provide some experimental parameters to produce these phenomena that may be realized in future experiments. These are interesting results with potential physical implications, such as the formations of matter-wave bright solitons in BECs.

2. Theoretical model and solution method

In the present work, we consider a cigar-shaped BEC of a relatively low density, when the energy of two-body interactions is much less than the kinetic energy in the transverse direction ($\mathcal{N}|a_s| \ll a_\perp$, where $\mathcal{N}$ is a total number of atoms, a_s is the s-wave scattering length, and $a_\perp = \sqrt{\hbar/m\omega_\perp}$ is the linear oscillator lengths in the transverse direction with $\omega_\perp$ being the transverse trapping frequency). The macroscopic wave function of a quasi-one-dimensional BEC can be written as the following dimensionless form [15]

$$i\psi_t = -\psi_{xx} + v(x,t)\psi + g(x,t)|\psi|^2\psi. \tag{1}$$

Here time t and coordinate x are measured in units $2/\omega_\perp$ and $a_\perp$, respectively. The trapping potential we consider here is assumed to be $v(x,t) = -\theta(t)^2x^2$ with $\theta = |\omega_0|/\omega_\perp \ll 1$, where ω_0 denotes the axial-oscillation frequency. The nonlinear coefficient reads $g(x,t) = 4|a_s(x,t)|/a_B$ (a_B is the Bohr radius) controlled by the FR, in which the scattering length $a_s(x,t)$ is a space- and/or time-modulated function of the varying magnetic field $B(x,t)$. In real experiments, the spatially dependent magnetic field may be generated by a microfabricated ferromagnetic structure integrated on an atom chip [42]. A constant θ^2 implies an oscillator potential which can be confining or expulsive for $\theta^2 < 0$ or $\theta^2 > 0$, respectively.

Eq. (1) also finds the applications in nonlinear optics. In this case, t and x, respectively, denote the propagation distance and the retarded time, $\psi(x,t)$ is the complex envelope of the electric field, $v(x,t)$ describes the contribution to the refractive index, and $g(x,t)$ represents the Kerr coefficient which can be achieved by the nonuniform distribution in waveguide.

The normalized number of atoms reads

$$N = \int_{-\infty}^{\infty} |\psi|^2\,\mathrm{d}x, \tag{2}$$

which is connected with $\mathcal{N}$ via the relation $\mathcal{N} = a_\perp N/(2a_B)$. In terms of the optical-beam transmission, N is proportional to the total power of the trapped light signal. The energy of Eq. (1) can be written as

$$\mathcal{E} = \int_{-\infty}^{\infty} \left[\left|\frac{\partial\psi}{\partial x}\right|^2 + v(x,t)|\psi|^2 + \frac{g(x,t)}{2}|\psi|^4\right]\mathrm{d}x. \tag{3}$$

By following the scheme proposed in Ref. [15], exact solutions can be constructed by casting Eq. (1) into the form of a solvable stationary NLSE

$$E\Phi = -\Phi_{XX} + G|\Phi|^2\Phi, \tag{4}$$

which may be implemented by employing the following transformation:

$$\psi(x,t) = \sqrt{\gamma}Z(\xi)e^{i\varphi(x,t)}\Phi\big[X(x,t)\big], \tag{5}$$

where $X(x,t) = \int_{-\infty}^{\xi} Z[\xi'(x,t)]^{-2}\,\mathrm{d}\xi'$ with $\xi(x,t) = \gamma(t)x + p\int_0^t \gamma(t)^2\,\mathrm{d}t$. The constant E is the corresponding eigenvalue (which corresponds to the chemical potential in BEC and the propagation constant in nonlinear optics), G determines the sign of nonlinearity, and p controls the movement behavior of solution. Time-dependent function $\gamma(t)$ is the inverse of width of the localized solution.

This transformation requires

$$g(x,t) = G\gamma Z(\xi)^{-6}, \tag{6}$$

$$\varphi(x,t) = -\frac{\gamma_t}{4\gamma}x^2 - \frac{p\gamma}{2}x + \left(1 - \frac{p^2}{4}\right)\int_0^t \gamma^2\,\mathrm{d}t. \tag{7}$$

Defining $\chi(t) = 1/\gamma(t)$, one can find that $\chi(t)$ and $Z(\xi)$, respectively, satisfy the following Mathieu and Ermakov–Pinney equations [43,44]

$$\chi_{tt} - 4\theta(t)^2\chi = 0, \tag{8}$$

$$Z_{\xi\xi} - Z = E/Z^3. \tag{9}$$

For Mathieu equation (8), the solutions may be found analytically, depending on the choice of θ. In particular, when $\theta = 0$, the potential is vanishing and $\chi(t) = \alpha_0 t + \beta_0$, where α_0 and β_0 are real constants. Note that, in this case, the soliton solutions can be found analytically if constants α_0 and β_0 are chosen properly. Since the main purpose of this work is to study the properties of some special localized solution structures, i.e., formations of n-order TSBSs, we set $\theta \neq 0$.

The solution of Ermakov–Pinney equation (9) can be constructed as

$$Z(\xi) = \sqrt{A\phi_1^2 + 2B\phi_1\phi_2 + C\phi_2^2}, \tag{10}$$

where A, B and C are real constants satisfying $E = (AC - B^2)W^2$, and the constant Wronskian $W = \phi_1\phi_{2\xi} - \phi_2\phi_{1\xi}$ with $\phi_1(\xi)$ and $\phi_2(\xi)$ being two linearly independent solutions of $\phi_{\xi\xi} - \phi = 0$. It is easy to obtain that: $\phi_1(\xi) = e^{\xi}$ and $\phi_2(\xi) = e^{-\xi}$. Since $\phi_1(0) = \phi_{1\xi}(0) = \phi_2(0) = 1$, and $\phi_{2\xi}(0) = -1$, the Wronskian is $W = -2$.

Next, we will show the formations of TSBSs in harmonic potentials for some physical relevant choice of θ. To do this, we choose $p = 0$ in the following discussions. With this, the phase given by Eq. (7) has the quadratic nature.

3. Formations of *n*-order TSBSs in BECs for attractive nonlinearity

Concerning the solutions of Eq. (4) for $E = 0$. In such a case, a nontrivial exact solution is given by $\Phi(X) = \lambda/\sqrt{-G}\,\mathrm{cn}(\lambda X -$