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Dynamics of analytical three-dimensional matter-wave solutions in Bose–Einstein condensates with multi-body interactions

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ABSTRACT

Using the F-expansion method we obtain a class of analytical matter-wave solutions to Bose–Einstein condensates with multi-body interactions through the three-dimensional quintic Gross–Pitaevskii equation. Our results demonstrate that the dynamics of matter-wave solutions can be controlled by selecting the potential, quintic nonlinearity, and gain coefficients. The obtained matter-wave solutions may be generated by tuning the cubic nonlinearity to zero via the Feschbach resonance technique and making the quintic nonlinearity increasing rapidly enough toward the periphery. The stability analysis of the obtained matter-wave solutions is investigated analytically and numerically.

1. Introduction

The Gross–Pitaevskii equation (GPE) and its variants are the most useful physical models in Bose–Einstein condensates (BECs), where it describes the behavior of the condensate wave function [1]. Various types of solutions to GPE, such as bright (dark) solitons [2], periodic traveling waves [3], and localized waves [4,5], are found of great interest because of their applications to a diverse array of physical systems. As a general form, the inverse problem method, which is responsible for the existence and stability of solitons, is considered as the effective technique to solve the GPE model [6]. A very important aspect of the GPE is the stability of its solutions; that is, how do they evolve in time when disturbed from their analytically given forms, which can be addressed numerically. It is expected that the stability of multidimensional solutions will be enhanced in GPE models with oscillating dispersion/diffraction and/or sign-changing nonlinearity [7], or, can conveniently be addressed by the dispersion and nonlinearity management methods [8]. The possibility to stabilize multidimensional solitons has also been reported in Refs. [9–12].

Usually, the nonlinear interactions in BEC are of a cubic nature. However, the cubic-quintic (CQ) nonlinearity can occur when the two- and three-body interactions are considered. In this case, the properties of the BECs could be significantly affected by the multi-body interactions characterized by the s-wave scattering length a_s, which is controlled by the Feschbach resonance (FR) techniques [13]. Moreover, if the interactions of atomic cloud are considered as well, the governing equation should still include the gain (loss) term. In these regimes, a more accurate treatment of the mean-field energetics of the dense condensates will need to account for both two- and three-body elastic collisions [14]. According to the zero-temperature mean-field theory, dynamics of the three-dimensional BEC with multi-body interactions satisfies the following GP equation [15]

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\Delta\psi + V(\mathbf{r},t)\psi + f\left(|\psi|^2\right)\psi \tag{1}$$

for the condensate wave-function ψ, where $V(\mathbf{r},t)$ is the potential of the external forces trapping the condensate. The contribution of multi-particle collisions has a complicated form of the nonlinear term

$$f\left(|\psi|^2\right) = g|\psi|^2 + \chi|\psi|^4, \tag{2}$$

where g and χ are the nonlinear coefficients corresponding to the two- and three-body interactions, respectively. In BECs, the cubic nonlinearity coefficient g can vanish, which results in the quintic GPE (QGPE). This model can be derived by setting the s-wave scattering length a_s to zero via the FR technique [15,16]. In this case, the three-body collisions could have an appreciable contribution even in a very dilute regime, when the so-called Efimov

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effect [17] becomes possible and the two-body scattering length becomes larger than the effective two-body interaction radius. For this, a very large number of three-body bound states (so-called Efimov states) can be formed and the contribution of the three-body elastic collisions to the density energy may become comparable to the one arising from two-body interactions. Mostly important, one can change the strength and the sign of the three-body interaction by controlling the strength of the two-body interaction via the FR. Theoretical investigations of the three-body interactions associated with the QGPE were provided in [18,19].

Based on the above considerations, in this work we construct analytical matter-wave solutions of the QGPE with quadratic potential and a gain (loss) term:

$$i\psi_t = -\frac{1}{2}\Delta\psi + V(\mathbf{r},t)\psi + \chi|\psi|^4\psi + i\gamma\psi, \tag{3}$$

where $\psi(x,y,z,t)$ is the normalized wave function of the condensate with $N=\int|\psi|^2\,\mathrm{d}\mathbf{r}$ being the number of atoms. Here t is the time, $\Delta = \partial_x^2+\partial_y^2+\partial_z^2$ represents the 3D Laplace operator, and $r=\sqrt{x^2+y^2+z^2}$ is the position coordinate. $V(\mathbf{r},t)=\alpha(t)r^2$ is a 3D isotropic harmonic potential [20] with $\alpha(t)$ being its strength. When one controls the dynamics of BEC in the trap [21], the trapping frequency can be a function of time t, which leads to the strength of the harmonic potential $\alpha(t)$ varying with time t. The parameter γ is the gain or loss coefficient, which is phenomenologically incorporated to account for the interaction of atomic or thermal clouds. The QGPE also appears in general nonlinear Schrödinger-type systems near the transition from supercritical to subcritical bifurcations [22], pattern formation [23], and dissipative solitons [24].

The paper is organized as follows. In Section 2, the solution method to Eq. (3) is presented. In Section 3, some analytical matter-wave solutions are obtained, which may be generated by setting the s-wave scattering length a_s to zero via the FR technique and making the quintic nonlinearity increasing rapidly enough toward the periphery. In Section 4, the stability analysis of the obtained matter-wave solutions is investigated analytically and numerically. Finally, the main findings are summarized in Section 5.

2. Solution method

Utilizing the F-expansion technique and the balance principle [11], we can write the complex wave function ψ in terms of its amplitude and phase:

$$\psi(x,y,z,t) = A(x,y,z,t)\exp\left[iB(x,y,z,t)\right]. \tag{4}$$

Substituting (4) into Eq. (3), one finds the following coupled equations:

$$A_t + \frac{1}{2}\left[2(A_xB_x + A_yB_y + A_zB_z) + A\Delta B\right] = \gamma A, \tag{5}$$

$$-AB_t + \frac{1}{2}\left[\Delta A - A\left(B_x^2+B_y^2+B_z^2\right)\right] - \alpha r^2 A - \chi A^5 = 0. \tag{6}$$

Next, we assume

$$A = f(t)\sqrt{F(\theta)} + h(t)\sqrt{F^{-1}(\theta)}, \tag{7}$$

$$\theta = k(t)x + l(t)y + m(t)z + \omega(t), \tag{8}$$

$$B = a(t)r^2 + b(t)(x+y+z) + e(t), \tag{9}$$

where f,h,k,l,m,ω,a,b,e are real functions of t to be determined, and F is a Jacobi elliptic function (JEF), which satisfy the following general first and second-order nonlinear ordinary differential equations: $(\frac{\mathrm{d}F}{\mathrm{d}\theta})^2 = c_0 + c_2F^2 + c_4F^4$, and $\frac{\mathrm{d}^2F}{\mathrm{d}\theta^2} = c_2F + 2c_4F^3$,

Table 1
JEFs. When $0<M<1$, the JEFs are periodic traveling wave solutions. When $M\to 0$, the periodic traveling wave solutions evolve into the periodic trigonometric functions. When $M\to 1$, the periodic traveling wave solutions become the time-dependent soliton solutions. When $M=0$ or 1, only some of the functions may be utilized, because of the developing singularities.

	c_0	c_2	c_4	F	$M=0$	$M=1$
1	1	$-(1+M^2)$	M^2	sn	sin	tanh
2	$1-M^2$	$2M^2-1$	$-M^2$	cn	cos	sech
3	M^2-1	$2-M^2$	-1	dn	1	sech
4	M^2	$-(1+M^2)$	1	ns	csc	coth
5	$-M^2$	$2M^2-1$	$1-M^2$	nc	sec	cosh
6	-1	$2-M^2$	M^2-1	nd	1	cosh
7	1	$2-M^2$	$1-M^2$	sc	tan	sinh
8	$1-M^2$	$2-M^2$	1	cs	cot	csch
9	1	$-(1+M^2)$	M^2	cd	cos	1
10	M^2	$-(1+M^2)$	1	dc	sec	1

where c_0, c_2, and c_4 are real constants related to the elliptic modulus M of JEFs (see Table 1).

Substituting Eqs. (7)–(9) into Eqs. (5) and (6) and requiring that $x^jF^{\pm n/2}, y^jF^{\pm n/2}, z^jF^{\pm n/2}$ ($j=0,1,2$; $n=0,1,2,3,4,5$), and $\sqrt{c_0+c_2F^2+c_4F^4}$ of each term be separately equal to zero, we obtain a system of equations:

$$\frac{\mathrm{d}\Omega_i}{\mathrm{d}t} + 3a\Omega_i - \gamma\Omega_i = 0, \tag{10}$$

$$\frac{\mathrm{d}S}{\mathrm{d}t} + 2Sa = 0, \tag{11}$$

$$\frac{\mathrm{d}\omega}{\mathrm{d}t} + b(k+l+m) = 0, \tag{12}$$

$$\frac{\mathrm{d}a}{\mathrm{d}t} + 2a^2 + \alpha = 0, \tag{13}$$

$$f\left[\frac{3}{8}(k^2+l^2+m^2)c_4 - \chi f^4\right] = 0, \tag{14}$$

$$h\left[\frac{3}{8}(k^2+l^2+m^2)c_0 - \chi h^4\right] = 0, \tag{15}$$

$$\frac{\mathrm{d}e}{\mathrm{d}t} + \frac{3}{2}b^2 - \frac{1}{8}(k^2+l^2+m^2)c_2 + 10\chi f^2h^2 = 0, \tag{16}$$

$$\frac{1}{8}h(k^2+l^2+m^2)c_4 + 5\chi f^4h = 0, \tag{17}$$

$$\frac{1}{8}f(k^2+l^2+m^2)c_0 + 5\chi fh^4 = 0, \tag{18}$$

where $\Omega_i = f,h$. From the above equations one can see that the analytical solution of this system can be found only if Riccati-type Eq. (13) for the parameter function $a(t)$ can be solved analytically. All other parameters depend on $a(t)$ explicitly or implicitly.

Introducing a single auxiliary function $\delta(t)=\int_0^t a\,\mathrm{d}t$, we obtain the following solutions by solving Eqs. (10)–(18):

$$S = S_0\exp(-2\delta), \tag{19}$$

$$f = f_0\exp(-3\delta)\exp\left(\int_0^t \gamma\,\mathrm{d}t\right), \qquad h = \epsilon\left(\frac{c_0}{c_4}\right)^{\frac{1}{4}} f, \tag{20}$$

$$\omega = \omega_0 - b_0(k_0+l_0+m_0)\int_0^t \exp(-4\delta)\,\mathrm{d}t, \tag{21}$$

$$e = e_0 + \frac{1}{8}\left[\left(c_2 + 18\epsilon^2\sqrt{c_0c_4}\right)\left(k_0^2+l_0^2+m_0^2\right) - 12b_0^2\right] \times \int_0^t \exp(-4\delta)\,\mathrm{d}t, \tag{22}$$