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Nonlinear fast growth of water waves under wind forcing

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ABSTRACT

In the wind-driven wave regime, the Miles mechanism gives an estimate of the growth rate of the waves under the effect of wind. We consider the case where this growth rate, normalised with respect to the frequency of the carrier wave, is of the order of the wave steepness. Using the method of multiple scales, we calculate the terms which appear in the nonlinear Schrödinger (NLS) equation in this regime of fast-growing waves. We define a coordinate transformation which maps the forced NLS equation into the standard NLS with constant coefficients, that has a number of known analytical soliton solutions. Among these solutions, the Peregrine and the Akhmediev solitons show an enhancement of both their lifetime and maximum amplitude which is in qualitative agreement with the results of tank experiments and numerical simulations of dispersive focusing under the action of wind.

1. Introduction

The investigation of the physical mechanisms for the generation of ocean waves by wind has a long history which starts at the beginning of the 20th century [1] and is still ongoing. The problem is highly nonlinear [2] and the feedback at the air–water interface between wind and water waves is difficult to study experimentally and theoretically because of turbulence in both fluids.

The problem can be simplified at first by neglecting currents in the water and by considering the so-called wind-driven wave regime which is characterised by growing seas with wave ages $c_p/u^* < 30$, where c_p is the phase velocity of the water waves and u^* is the friction velocity of wind over water waves [3]. Direct field measurements of the pressure induced by airflow on waves are rare, thus there is no agreement in the scientific community on the underlying mechanisms leading to wave amplification (for a review see [2, Chapter 3] and [3]).

In the shear flow model introduced by Miles [4,5] the rate of energy transfer from the wind to a wave propagating at phase velocity c_p is proportional to the wind profile curvature $U''(z_c)$ at the critical height z_c where the wind speed equals the phase velocity of the wave, $U(z_c) = c_p$. The Miles mechanism has been recently confirmed in field experiments, in particular for long waves [6]. For a logarithmic velocity profile in the boundary layer, the Miles growth rate Γ_M results in [4,7,8]

$$\frac{\Gamma_M}{\omega} = \frac{\Gamma_M}{2\pi f} \equiv \frac{1}{\omega E}\frac{dE}{dt} = \delta\alpha\left(\frac{u^*}{c_p}\right)^2 \tag{1}$$

where E is the wave energy, f is the frequency of the carrier wave, $\delta = \rho_a/\rho_w$ is the density ratio (1.29×10^{-3} between air and water), and α is an empirical constant of the order of 32.5 in the wind-driven wave regime [7]. The pressure P induced at the water surface then depends on the surface elevation η as follows [4,8]

$$\frac{1}{\rho_w} P(x,t) = \frac{\Gamma_M}{f}\frac{c_p^2}{2\pi}\eta_x(x,t) \tag{2}$$

Typical values of Γ_M/f are shown in Fig. 1 of [7] (in that figure $\Gamma_M = \gamma$) or in Fig. 1 of [9] (where $\Gamma_M = \beta$) as a function of the wave age c_p/u^*. They range from 10^{-3}–10^{-2} for fast-moving waves ($c_p/u^* > 5$) to 10^{-2}–1 for slow-moving waves and laboratory tank experiments ($c_p/u^* \leqslant 5$). Thus, the growth rate can be regarded as a small parameter in the wind-driven wave regime and generally it is assumed that $\Gamma_M/f = O(\epsilon^2)$, where $\epsilon = ak$ is the wave steepness, a being the amplitude of the vertical water displacement η and k the wavenumber of the water wave. For weak-nonlinear waves the steepness is indeed small and in ocean waves it is smaller than 0.55, the value for which wave-breaking occurs [10]. The case $\Gamma_M/f = O(\epsilon^2)$ gives rise to the following damped/forced nonlinear Schrödinger equation [11,8,12]

$$i\frac{\partial a}{\partial t} - \frac{1}{8}\frac{\omega}{k^2}\frac{\partial^2 a}{\partial x^2} - \frac{1}{2}\omega k^2|a|^2 a = i\left(\frac{\Gamma_M}{2} - 2\nu k^2\right)a \tag{3}$$

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where ν is the kinematic viscosity. Thus, the case $\Gamma_M/f = O(\epsilon^2)$ describes the quasi-equilibrium between wind and damping effects due to viscosity.

In this Letter, we consider the case $\Gamma_M/f = O(\epsilon)$, corresponding to stronger winds, the effect of which overcomes the dissipation due to viscosity. This case turns out to be relevant for explaining experimental results obtained in the context of dispersive focusing of waves under the action of wind [13,14]. We will insert the aerodynamic pressure term, given in Eq. (2), into the Bernoulli equation evaluated at the ocean surface and we will use the method of multiple scales to obtain the corresponding nonlinear Schrödinger equation in the case of fast-growing waves. Due to the universality of the NLS equation in many other fields of physics, the considered case can in principle be of interest in other physical situations where the multiple-scale method can be applied and the forcing term is introduced at first order in the development parameter.

2. Governing equations and the method of multiple scales (MMS)

We recall here the equations governing the propagation of surface gravity waves in the presence of wind and the main assumptions used in the method of multiple scales for deriving the NLS equation.

At low viscosity the water-wave problem can be set within the framework of potential flow theory [15] and the two-dimensional flow of a viscous, incompressible fluid is governed by the Laplace equation

$$\nabla^2\phi = 0 \tag{4}$$

where $\phi(x,z)$ is the velocity potential. This equation is solved together with the kinematic boundary condition at the free surface $\eta(x,t)$

$$\eta_t + \phi_x\eta_x - \phi_z = 2\nu\eta_{xx} \quad \text{at } z = \eta(x,t) \tag{5}$$

and at the bottom

$$\phi_z = 0 \quad \text{at } z = -H \tag{6}$$

The other boundary condition is given by the Bernoulli equation which at the free surface takes the form

$$\phi_t + \frac{1}{2}\left(\phi_x^2 + \phi_z^2\right) + g\eta = -\frac{P}{\rho_w} - 2\nu\phi_{zz} \quad \text{at } z = \eta(x,t) \tag{7}$$

where g is the gravity acceleration and P is the excess pressure at the ocean surface in the presence of wind, given by Eq. (2) in the context of the Miles mechanism.

We use the method of multiple scales (MMS) to find the terms in the NLS equations which are related to the wind forcing with a growth rate of first order in the wave steepness, $\Gamma_M/f = O(\epsilon)$. This method is based on the fact that temporal and spatial scales of the carrier wave $(1/\omega, 1/k)$ are much smaller than those of the envelope. MMS has been used for deriving the NLS equation under the assumption of small nonlinearity, $\epsilon = ak \ll 1$, and narrow spectral width $\Delta k/k \ll 1$ [16] and successfully applied for including high-order nonlinear terms [17] and constant vorticity in water waves [18], or in other physical contexts. For example, in the context of the propagation of optical waves in nonlinear materials [19], this method is also known as the slowly varying envelope approximation (SVEA) [20,21].

The velocity potential ϕ and the surface elevation η have the following representations [17,18]

$$\phi = \sum_{j=1}^{\infty}\epsilon^j\phi_j, \quad \phi_j = \phi_{j0} + \sum_{n=1}^{j}\phi_{jn}\mathcal{E}^n + \text{c.c.} \tag{8}$$

$$\eta = \sum_{j=1}^{\infty}\epsilon^j\eta_j, \quad \eta_j = \eta_{j0} + \sum_{n=1}^{j}\eta_{jn}\mathcal{E}^n + \text{c.c.} \tag{9}$$

where the second index in the amplitudes ϕ_{jn}, η_{jn} refers to the harmonics

$$\mathcal{E}^n = \frac{1}{2}\exp\left[in(kx_0 - \omega t_0)\right] \tag{10}$$

The velocity potential at the free surface, $\phi(x, z = \eta, t)$, is written as a Taylor expansion around $z = 0$:

$$\phi(x,\eta,t) = \sum_{j=0}^{+\infty}\frac{\eta_j}{j!}\partial_z^j\phi\bigg|_{z=0} \tag{11}$$

The operators for the derivatives are replaced by sums of operators

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial t_0} + \epsilon\frac{\partial}{\partial t_1} + \epsilon^2\frac{\partial}{\partial t_2} + \cdots \tag{12}$$

corresponding to fast and slow temporal derivatives, and analogously for $\partial/\partial x$. We use the same notation as in Ref. [18] (note however that the order of indices in the amplitudes ϕ_{jn}, η_{jn} is inverted).

3. Wind-forced NLS equation

In this section we apply the method of multiple scales for developing the governing equations in terms of the expansion parameter ϵ. Terms of linear order in ϵ give the dispersion relation $\omega = \sqrt{g\sigma k}$, where $\sigma = \tanh(kH)$, and they are not affected by wind forcing. The wind forcing terms appear in the expansion at second order in the following relations:

$$\frac{\partial A}{\partial t_1} + c_g\frac{\partial A}{\partial x_1} = \Gamma_M\frac{\sigma A}{2} \tag{13}$$

$$\eta_{21} = \frac{1}{g}\left(i\omega D + c_g\frac{\partial A}{\partial x_1} + \Gamma_M\frac{\sigma A}{2}\right) \tag{14}$$

where $A = \phi_{11}|_{z=0}$ and $D = \phi_{21}|_{z=0}$.

At third order, the new terms are $\partial\eta_{21}/\partial t_1$ in the kinetic boundary condition (5), which must be evaluated using Eq. (14), and $(\Gamma_M/\omega)c_p^2(\partial\eta_{11}/\partial x_1 + \partial\eta_{21}/\partial x_0)$ in the Bernoulli equation at $z = 0$, Eq. (7). Including these terms finally gives the wind-forced NLS equation in the limit of deep-water waves, $kH \gg 1$:

$$i\frac{\partial a}{\partial t_2} - \beta_1\frac{\partial^2 a}{\partial x_1^2} - \beta_2\frac{\partial a}{\partial x_1} - \beta_3 a - Ma|a|^2 = -2i\nu k^2 a \tag{15}$$

where $a = 2iA/c_p$, $\beta_1 = -(dc_g/dk)/2 = \omega/(8k^2)$, $\beta_2 = \Gamma_M(\omega + c_g k)/(2\omega k) = 3\Gamma_M/(4k)$, $\beta_3 = \Gamma_M^2/(8\omega)$ and $M = \omega k^2/2$.

Note that the equation that we obtain for $\Gamma_M/f = O(\epsilon)$ differs, as it should, from the usual equation obtained assuming $\Gamma_M/f = O(\epsilon^2)$. When $\Gamma_M/f = O(\epsilon^2)$, Eq. (15) reduces to Eq. (3). Indeed, the terms proportional to Γ_M in Eqs. (13)–(14) become of higher order. Moreover, in the Bernoulli equation at $z = 0$, the term $(\Gamma_M/\omega)c_p^2(\partial\eta_{11}/\partial x_1 + \partial\eta_{21}/\partial x_0)$ becomes $(\Gamma_M/\omega)c_p^2\partial\eta_{11}/\partial x_0$. This term in the forced NLS equation reduces for $\Gamma_M/f = O(\epsilon^2)$ to the term $-i\Gamma_M a/2$, which corresponds to the one on the right-hand side of Eq. (3). As we will see in the next section, the two terms in the NLS equation (15) due to wind forcing correspond to a variation of the dispersion term and of the phase of the wave field.

It is interesting to calculate the energy evolution. The Miles growth rate is recovered from the relations at second-order expansion. Indeed, multiplying Eq. (13) by the complex conjugate a^*, adding the obtained equation to its complex conjugate and integrating by parts yields