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On the entropy of radiation reaction

David A. Burton [a,b,*], Adam Noble [c]

[a] Department of Physics, Lancaster University, Lancaster, LA1 4YB, United Kingdom
[b] Cockcroft Institute of Accelerator Science and Technology, Daresbury, WA4 4AD, United Kingdom
[c] Department of Physics, SUPA and University of Strathclyde, Glasgow, G4 0NG, United Kingdom

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ABSTRACT

The inexorable development of ever more powerful laser systems has re-ignited interest in electromagnetic radiation reaction and its significance for the collective behavior of charged matter interacting with intense electromagnetic fields. The classical radiation reaction force on a point electron is non-conservative, and this has led some authors to question the validity of methods used to model ultra-intense laser–matter interactions including radiation reaction. We explain why such concern is unwarranted.

1. Introduction

Contemporary advances in ultra-intense laser facilities have driven the recent surge of interest in the collective behavior of charged matter in extreme conditions, and a particularly fascinating topic in that context concerns the coupling of an electron to its own radiation field [1]. An accelerating electron emits electromagnetic radiation, and the energy and momentum carried away by the electromagnetic field must be properly accommodated. In most practical cases, the Lorentz force on an electron due to an applied electromagnetic field is considerably larger than the force due to the electron's emission, and the effect of the recoil due to the emitted radiation is negligible or can be adequately represented using simple physical reasoning. Although such arguments avoid the difficulties that plague more comprehensive analyses, the parameter regimes promised by forthcoming ultra-intense laser facilities ensure that more fundamental considerations are now of practical necessity. For example, ELI [2] is expected to operate with intensities 10^{23} W/cm^2 and electron energies in the GeV range, at which level the radiation reaction force becomes comparable to, and can even exceed, the applied force due to the laser field.

Several experiments have been proposed in recent years to test the effects of radiation reaction in ultra-intense laser–matter interactions (see Ref. [3] for a recent review). Particular attention has been paid to the behavior of a bunch of classical electrons driven by an ultra-intense laser pulse where the forces between the electrons are negligible compared to the forces exerted by the laser pulse. Fortunately, the Landau–Lifshitz equation [4] for a single classical electron is integrable when the pulse is modeled as a plane wave [5,6] and the computational advantages afforded by neglecting interactions between the electrons are considerable. Theoretical investigation of the collision between an electron and a laser pulse has shown that the electron can reverse its direction of motion if radiation reaction is taken into account [7,8] and it may be possible to detect this effect in the radiation spectrum. In addition, it has been shown [9,10] that the volume of the region of phase space occupied by a bunch of non-interacting electrons reduces with time (the bunch cools) due to radiation reaction in the ambient laser field.

However, the use of kinetic theory to describe a bunch of non-interacting classical point electrons in this context has recently been criticized [11] because of the non-Hamiltonian nature of the Landau–Lifshitz equation (or its progenitor, the Lorentz–Dirac equation [12]). As a consequence, the entropy 4-current is not divergenceless in kinetic theories induced from the Landau–Lifshitz equation [13,9,10] or from the Lorentz–Dirac equation [14].

Furthermore, inter-particle interactions should not be ignored in all situations where radiation reaction plays a role. If the heating due to the stochasticity of photon emission [15] and the discreteness of charge [10] within the bunch can be neglected, one might anticipate that the recoil due to emission of radiation would cool the bunch of electrons in all situations. However, we recently showed [14] that inter-particle interactions may heat the bunch. This Letter explores the significance of this observation, and the pathway that we tread leads directly to an explanation of why the recent criticisms given in Ref. [11] are unjustified.

* Corresponding author.

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2. Non-relativistic considerations

The simplest way to quickly obtain a flavor of the effects of inter-particle interactions is to consider the behavior of a bunch of *non-relativistic* electrons, and assume that the inter-particle forces due to the magnetic fields they generate may be neglected. The force on an electron in the bunch is a superposition of the Lorentz forces exerted by the other electrons in the bunch and the force on the electron due to its own radiation field. For simplicity, we neglect collisions between the electrons and represent the inter-particle forces using a mean field approximation $\boldsymbol{E}$ to their electric field.

The Abraham–Lorentz equation (see, for example, Ref. [1])

$$m\frac{d^2\boldsymbol{x}}{dt^2} = q\boldsymbol{E}(\boldsymbol{x},t) + m\tau\frac{d^3\boldsymbol{x}}{dt^3} \tag{1}$$

determines the position $\boldsymbol{x}(t)$ of a non-relativistic electron in an ambient smooth electric field $\boldsymbol{E}$, where m is the mass of the electron, $q=-e$ is the charge on the electron and the time constant $\tau = q^2/6\pi\epsilon_0 mc^3 = 2r_e/3c$ where r_e is the classical radius of the electron. The total force on the electron is the sum of the mean field approximation $q\boldsymbol{E}$ to the total force exerted by other electrons in the bunch and the reaction $m\tau\, d^3\boldsymbol{x}/dt^3$ due to the electron's own emission. From now on, we will reserve the term *bunch* for the smooth continuum specified by the charge density $\epsilon_0\nabla\cdot\boldsymbol{E}$.

Following the iterative procedure introduced by Landau and Lifshitz [4], the introduction of the requirement $m\,d^3\boldsymbol{x}/dt^3 = q\dot{\boldsymbol{E}} + \mathcal{O}(\tau)$ removes runaway solutions and (1) can be written as

$$m\frac{d^2\boldsymbol{x}}{dt^2} = q\boldsymbol{E}(\boldsymbol{x},t) + q\tau\left[\partial_t\boldsymbol{E}(\boldsymbol{x},t) + \left(\frac{d\boldsymbol{x}}{dt}\cdot\nabla\right)\boldsymbol{E}(\boldsymbol{x},t)\right] \tag{2}$$

where $\mathcal{O}(\tau^2)$ terms have been dropped and an overdot indicates d/dt.

Suppose that the initial position and velocity of the electron are sampled from a statistical ensemble of initial conditions, and let $\langle\boldsymbol{x}(t)\rangle$ be the ensemble average of the electron's position at time t. Introducing $\boldsymbol{x} = \langle\boldsymbol{x}\rangle + \boldsymbol{\xi}$ into the expansion of (2) to leading order in the random variable $\boldsymbol{\xi}$ leads to

$$m\frac{d^2\langle\boldsymbol{x}\rangle}{dt^2} = q\boldsymbol{E}\big(\langle\boldsymbol{x}\rangle,t\big) + q\tau\left[\partial_t\boldsymbol{E}\big(\langle\boldsymbol{x}\rangle,t\big) + \left(\frac{d\langle\boldsymbol{x}\rangle}{dt}\cdot\nabla\right)\boldsymbol{E}\big(\langle\boldsymbol{x}\rangle,t\big)\right] \tag{3}$$

and

$$\frac{d}{dt}\left(\frac{1}{2}m\langle\dot{\boldsymbol{\xi}}\cdot\dot{\boldsymbol{\xi}}\rangle\right) = \big\{q\langle\dot{\xi}^\mu\xi^\nu\rangle\partial_\nu E_\mu + q\tau\big[\langle\dot{\xi}^\mu\xi^\nu\rangle\partial_\nu\partial_t E_\mu + \langle\dot{\xi}^\mu\dot{\xi}^\nu\rangle\partial_\nu E_\mu + \langle\dot{x}^\nu\rangle\partial_\omega\partial_\nu E_\mu\langle\dot{\xi}^\mu\xi^\omega\rangle\big]\big\}\big|_{\boldsymbol{x}=\langle\boldsymbol{x}\rangle} \tag{4}$$

where Greek indices range over $1,2,3$ and the explicit time dependence of the electric field $\boldsymbol{E}$ in (4) has been suppressed for notational convenience.

Simple choices for $\langle\dot{\xi}^\mu\xi^\nu\rangle|_{t=0}$ and $\langle\dot{\xi}^\mu\dot{\xi}^\nu\rangle|_{t=0}$ reveal the significance of (4). Suppose that the initial velocity and initial position of the electron are uncorrelated, and there is no preferred direction for its initial velocity. Hence $\langle\dot{\xi}^\mu\xi^\nu\rangle|_{t=0} = 0$ and $\langle\dot{\xi}^\mu\dot{\xi}^\nu\rangle|_{t=0} = \delta^{\mu\nu}\langle\dot{\boldsymbol{\xi}}\cdot\dot{\boldsymbol{\xi}}\rangle/3$, where $\delta^{\mu\nu}$ is the Kronecker delta, and using (4) it follows

$$\frac{d}{dt}\left(\frac{1}{2}m\langle\dot{\boldsymbol{\xi}}\cdot\dot{\boldsymbol{\xi}}\rangle\right)\bigg|_{t=0} = \left[q\tau\frac{1}{3}\langle\dot{\boldsymbol{\xi}}\cdot\dot{\boldsymbol{\xi}}\rangle\nabla\cdot\boldsymbol{E}\right]\bigg|_{\boldsymbol{x}=\langle\boldsymbol{x}\rangle,\,t=0}. \tag{5}$$

Let N electrons be represented by a small (finite) element of the bunch, where the element has volume V and the element's centroid is located at $\boldsymbol{x} = \langle\boldsymbol{x}\rangle$. Hence, the charge density ρ of the bunch and electric field $\boldsymbol{E}$ satisfy $\nabla\cdot\boldsymbol{E} = \rho/\epsilon_0$ with $\rho(\langle\boldsymbol{x}\rangle,t) = qN/V$.

If the initial velocities of the N electrons are described by a Maxwell–Boltzmann distribution (with temperature T), using (5) the thermal kinetic energy $U = N\frac{1}{2}m\langle\dot{\boldsymbol{\xi}}\cdot\dot{\boldsymbol{\xi}}\rangle$ of the N electrons satisfies

$$\frac{dU}{dt}\bigg|_{t=0} = \left[\tau\frac{k_B T}{m\epsilon_0}\rho^2 V\right]\bigg|_{\boldsymbol{x}=\langle\boldsymbol{x}\rangle,\,t=0} \tag{6}$$

where $\langle\dot{\boldsymbol{\xi}}\cdot\dot{\boldsymbol{\xi}}\rangle = 3k_B T/m$ has been used, with T the local temperature of the element. It follows from (6) that $dT/dt|_{t=0} > 0$ and the initial tendency of the element is to heat up, rather than cool down, due to radiation reaction. This result is surprising because we expect the bunch to cool in response to the emission of radiation.

Although the bunch is not in thermodynamic equilibrium, it is tempting to formally use the first law of thermodynamics $dU = T\,dS - p\,dV$ to introduce the entropy S of the element. The volume V of the element satisfies $dV/dt|_{t=0} = 0$ because $V \propto \langle\boldsymbol{\xi}\cdot\boldsymbol{\xi}\rangle^{3/2}$ and the initial position and velocity of each electron are uncorrelated. Hence, S satisfies

$$\frac{dS}{dt}\bigg|_{t=0} = \left[\tau\frac{k_B}{m\epsilon_0}\rho^2 V\right]\bigg|_{\boldsymbol{x}=\langle\boldsymbol{x}\rangle,\,t=0}. \tag{7}$$

The right-hand side of (7) is strictly positive, which is precisely how one expects the entropy of an isolated bunch of electrons to behave. However, more general considerations show that all is not as it seems.

3. Relativistic considerations

The Lorentz–Dirac equation is a fully relativistic description of a structureless point particle in an applied electromagnetic field F_{ab} and has the form

$$\frac{d^2x^a}{d\lambda^2} = -\frac{q}{m}F^a{}_b\frac{dx^b}{d\lambda} + \tau\Delta^a{}_b\frac{d^3x^b}{d\lambda^3} \tag{8}$$

with q the particle's charge, m the particle's rest mass, $\tau = q^2/6\pi m$ in Heaviside–Lorentz units with $c=\epsilon_0=\mu_0=1$, and the tensor $\Delta^a{}_b$ is

$$\Delta^a{}_b = \delta^a_b + \frac{dx^a}{d\lambda}\frac{dx_b}{d\lambda}. \tag{9}$$

For an electron, $q=-e<0$ as before. The Einstein summation convention is used throughout the following, indices are raised and lowered using the metric tensor $\eta_{ab} = \mathrm{diag}(-1,1,1,1)$ and lowercase Latin indices range over $0,1,2,3$. The particle's 4-velocity $dx^a/d\lambda$ is normalized as follows:

$$\frac{dx^a}{d\lambda}\frac{dx_a}{d\lambda} = -1 \tag{10}$$

where λ is the particle's proper time.

Dirac [12] derived (8) for a classical point electron by appealing to the conservation condition on the stress-energy–momentum tensor (see Ref. [16] for a recent discussion of the derivation). Dirac's approach required a regularization of the electron's singular contribution to the stress-energy–momentum tensor followed by a renormalization of the electron's rest mass. His procedure led to the third-order term in (8), which is the source of the famous pathological behavior exhibited by solutions to the Lorentz–Dirac equation (see Ref. [1], and also Ref. [17] for a recent discussion).

The standard approach to ameliorating the problems with the Lorentz–Dirac equation is to replace the third-order terms in (8)