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Numerical study of convection near the stability threshold in a square box with internal heat generation

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ABSTRACT

The Letter presents a numerical study of convection in a finite fluid layer heated from below and with homogeneous internal heat generation. Transitions between the conducting state, hexagonal, and roll flows are investigated for the Prandtl number in the range [0.1, 100] and the Rayleigh number from subcritical values up to $1.5\,Ra_{cr}$. The calculations reveal different directions of circulation in the stable hexagons above and below a critical Prandtl number value Pr_{cr}. Close to the Pr_{cr}, stable overcritical rolls are detected.

1. Introduction

This work investigates convection induced by internal heat generation. The phenomenon plays a fundamental role in many environmental and industrial processes, for example, motion in the atmosphere where heat is generated by the absorption of sunlight [1] and convection in the Earth's mantle where heat is produced by the radioactive decay of elements [2]. Internal heating also arises from chemical reactions and the application of an electric field to conducting fluids [3]. Convection retained by internal heat sources should be considered in nuclear safety engineering for predicting the behaviour of the nuclear reactor core cooling [4,5].

By means of direct numerical simulation, we investigate convection in a finite horizontal fluid layer heated from below and with uniform internal heating. The layer is confined between two rigid perfectly conducting horizontal plates kept at constant temperatures T_{bot} and T_{top}, with $T_{bot} > T_{top}$, and adiabatic side walls. The basic state of the fluid is static with the pressure in hydrostatic balance and a parabolic temperature profile between the upper and lower plates. This approach allows investigation of a great variety of questions concerning such problems as the onset of convection, flow evolution, stability of flow patterns, etc. A comparatively simple but realistic problem statement provides a possibility for experimental validation of the numerical results.

For the standard description in terms of the Oberbeck–Boussinesq equations, the system is determined by three dimensionless control parameters. Two of the parameters are essential for the classical Rayleigh–Bénard problem. These parameters are the Rayleigh number, $Ra = \frac{g\beta H^3 \delta T}{\nu\kappa}$, and the Prandtl number $Pr = \frac{\nu}{\kappa}$. The third parameter is induced by the internal heat generation, which can be expressed as $\bar{q} = \frac{QH^2}{\nu\delta T}$ [6]. Here g is the absolute value of the gravitational acceleration, $\beta = -\frac{1}{\rho}\frac{\partial\rho}{\partial T}$ is the thermal expansion coefficient, ρ is the fluid density, H is the thickness of the fluid layer, $\delta T = T_{bot} - T_{top}$ is the temperature difference between bottom and top plates, ν is the kinematic viscosity, κ is the thermal diffusivity, and Q is the volume strength of the heat source.

A nonlinear conductive profile relates an internal heating layer to other non-symmetric fluid arrangements caused, for example, by variable material properties (non-Boussinesq liquid), surface tension (Bénard–Marangoni convection), and different conditions on the upper and lower boundaries. Broken symmetry changes the scenario for the convection onset compared to the Rayleigh–Bénard convection in a layer with a linear conductive temperature profile. For the latter, an inherently symmetric case, the linear stability theory yields the critical Rayleigh number $\overline{Ra}_{cr}$ at which convection first sets in. For an infinite horizontal layer with rigid boundaries, $\overline{Ra}_{cr} = 1708$ [7]. Weakly nonlinear analysis declares that steady motion in the form of two-dimensional rolls is the only stable flow pattern immediately above the threshold. All three-dimensional patterns, such as hexagons or squares, are unstable [8]. Under non-symmetric conditions the critical Rayleigh number Ra_{cr} is less than in the symmetrical case, and steady motions are possible for Rayleigh numbers below the critical value prescribed by the linear theory [9,10]. This is finite amplitude subcritical convection, which can be induced by disturbances of sufficient amplitude. (Infinitesimal perturbations decay below Ra_{cr}.) Instead of

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two-dimensional rolls, hexagonal flow pattern becomes favoured in the vicinity of the threshold. As the Rayleigh number increases, hexagons lose their stability and transform into rolls [11].

The study F. Busse performed for non-Boussinesq systems [11] plays an important role in understanding convection onset in fluid arrangements with broken inversion symmetry. This researcher considered the temperature dependence of all relevant fluid properties: density, viscosity, thermal conductivity, and heat capacity. To describe the deviation from the Boussinesq approximation quantitatively, a special parameter, $\mathcal{P}$, has been introduced. The value of this parameter determines the shift in the critical Rayleigh number from the classical value $\overline{Ra}_{cr}$, and the stability boundaries for hexagons and rolls, the subcritical domain $Ra_{min}(\mathcal{P}, Pr) < Ra < Ra_{cr}$ where only finite amplitude motion is admissible. The sign of $\mathcal{P}$ defines the direction of circulation in hexagons. For $\mathcal{P} > 0$ stable hexagons have a downward flow in their centres, and for $\mathcal{P} < 0$ the flow is upward. When only thermal conductivity depends on temperature, $\mathcal{P}$ is proportional to $\partial\kappa/\partial T$ and has the same sign, that is, fluid in the centre of the cell flows in the direction of increasing heat conductivity. In a layer with rigid boundaries maintained at a fixed temperature, the conductive temperature profile due to $\partial\kappa/\partial T > 0$ is convex upward, as in a layer with homogeneous heating. The corresponding stable flow pattern is *down*-hexagons.

Theoretical and experimental studies of convective motion generated by internal heating began in the 1960s almost contemporaneously with the investigation of motion in non-Boussinesq liquids. However, the effect of internal heating has received considerably less attention than the effect of temperature dependent material properties, and from the beginning contradictory results were acquired. The discrepancies refer to the stability of hexagons with *up*- (*l*-type) and *down* (*g*-type) fluid motion in the centre of the cell, competition between rolls and hexagons as the preferred mode near the stability threshold, and the length of the subcritical domain. Qualitative validation of the results obtained for an internally heated layer has typically been performed through comparison with non-Boussinesq cases.

R. Krishnamurti used the similarity between different non-symmetric cases in her investigation of convection near the critical Rayleigh number in a horizontal layer with a steadily changing mean temperature [12,13]. Theoretical and experimental studies were performed for a horizontal layer bounded above and below by perfect heat conductors. The asymmetric configuration has been produced by changing the boundary temperature at a constant rate η. Homogeneous cooling at the boundaries results in the same mathematical problem as internal heating with a fixed temperature at the boundaries. Following F. Busse [11], stable finite amplitude motion in the form of hexagons was found to be the only stable planform in the subcritical domain $Ra_{min}(\eta) \leqslant Ra \leqslant Ra_{cr}(\eta)$, $R_{min} = Ra_{cr} - Ra_{min}$, and immediately above the critical value, at $Ra_{cr}(\eta) \leqslant Ra \leqslant Ra_{HR}(\eta)$, $R_{HR} = Ra_{HR} - Ra_{cr}$; $Ra_{min} \leqslant Ra \leqslant Ra_{HR}$ is the range where hexagons are the only stable flow pattern. Both studies [11,12] were performed for large Prandtl numbers. The direction of circulation in the stable hexagons was determined by the shape of the conductive temperature profile, but contrary to [11], hexagons with ascending flow in the centres were claimed to be stable as long as the conductive temperature profile was convex upward ($\eta < 0$). Recently, S.C. Generalis and F. Busse [14] shed light on this discrepancy testifying to the mistake in the algebraic transformation in [12].

Convection generated by internal heat sources has been investigated mostly in a layer with an isothermal, perfectly conducting upper boundary and an adiabatic lower boundary. In this arrangement, a hexagonal convection pattern with fluid descending in the centres of the cells has been predicted theoretically [15,16] and observed in experiments [17–19]. All of these studies have been performed for Rayleigh numbers above the critical value and for $Pr > 1$.

A theoretical investigation performed by M. Tveitereid and E. Palm [20] for an internally heated layer bounded by an isothermal plane above an insulating plane, deserves special attention. The authors demonstrated that the extent of the subcritical region Ra_{min}, being rather small, depends on Pr noticeably. For $Pr = 0.25$, $R_{min} = 0$ and everywhere else, $R_{min} > 0$; therefore, $Pr = 0.25$ appears to be a critical value (Pr_{cr}) for finite amplitude convection. Moreover, in the vicinity of Ra_{cr}, stable flow pattern consists of *down*-hexagons at $Pr > 0.25$, for $Pr < 0.25$ the stable planform is *up*-hexagons; close to $Pr = 0.25$ it is two-dimensional rolls. To the best of our knowledge, although predicted in [20], the exchange of stability between *down*-hexagons, rolls and *up*-hexagons, which occurs at varying Prandtl numbers and a Rayleigh number fixed at a value slightly above Ra_{cr}, has not been observed in experiments or in numerical simulations. This is also true for a fluid arrangement with isothermal upper and lower boundaries. In the latter case, there are no theoretical indications that different types of hexagons can be stable at moderate Prandtl numbers and $Pr \ll 1$. For example, theoretical study [14], concerned with the stability of secondary flows in a homogeneously heated layer with two rigid boundaries maintained at constant and equal temperatures, suggests stable finite amplitude motion in the form of *down*-hexagons both for $Pr = 0.1$ and $Pr = 7$ in the vicinity of Ra_{cr}.

Partially, the motivation for the study presented in this Letter is to clarify whether stable motion in an internally heated layer can exist in the form of *up*- and *down*-hexagons depending on Prandtl number. We study the effect of Rayleigh number, Prandtl number, and strength of internal heating on the convection onset, evolution of the flow structure and planform selection, transitions between the conducting state, roll flow and hexagonal flow with different directions of circulation in the cells.

2. Problem statement

The study is based on the 3D time-dependent Navier–Stokes equations in the Boussinesq approximation, the equation of continuity and the equation for temperature with a uniform heat source. The calculation domain is a rectangular box of height H and square in the horizontal direction $\overline{\Omega} = [0, l] \times [0, l] \times [0, H]$. The aspect ratio is $L = l/H = 15$. All boundaries of the box are rigid, the vertical walls are perfectly insulated, and the upper and lower boundaries are maintained at constant temperature T_{top} and T_{bot}, $T_{top} < T_{bot}$.

For a non-dimensional description, the length is scaled with the height H, $t_\nu = H^2/\nu$ and $\rho_0 \nu \chi / H^2$ are used as units of time and pressure, respectively, and ρ_0 is the liquid density at temperature T_{top}. Non-dimensional temperature is introduced as $T = (T_d - T_{top})/\delta T$, T_d is the dimensional temperature.

The governing equations in dimensionless form are written as [7, p. 18]:

$$\partial_t \mathbf{V} + (\mathbf{V} \cdot \nabla)\mathbf{V} = -\nabla p + \Delta \mathbf{V} + \frac{Ra}{Pr} T \mathbf{e}_z, \tag{1}$$

$$\nabla \cdot \mathbf{V} = 0, \tag{2}$$

$$\partial_t T + (\mathbf{V} \cdot \nabla) T = \frac{1}{Pr} \Delta T + \bar{q}. \tag{3}$$

Here $\partial_\xi \equiv \frac{\partial}{\partial \xi}$, $\xi = t, x, y, z$, $\nabla = (\partial_x, \partial_y, \partial_z)$, $\Delta = \nabla^2 = \partial^2_{xx} + \partial^2_{yy} + \partial^2_{zz}$, $(x, y, z) \in \Omega$, $\Omega = [0, L] \times [0, L] \times [0, 1]$, t denotes the time, $\mathbf{V} = (\mathbf{V}_x, \mathbf{V}_y, \mathbf{V}_z)$ the velocity, p the pressure, T the temperature, $\mathbf{e}_z = (0, 0, 1)$, $\bar{q}$ the heat source.

The basic state of pure conduction is described by the velocity $\mathbf{V} = 0$ and the temperature distribution $T_0(z) = 1 - z + z(1 - z)q/2$, where $q = \bar{q}\,Pr$ (Fig. 1).