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The Gravity Probe B experiment cannot sense spacetime torsion: on the Poincaré gauge theory of gravity and its equations of motion

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ABSTRACT

We discuss the structure of the Poincaré gauge theory of gravity (PG) that can be considered as the standard theory of gravity with torsion. We reconfirm that torsion, in the context of PG, couples only to the *elementary particle spin* and under no circumstances to the orbital angular momentum of test particles. We conclude that, unfortunately, the investigations of Mao et al. (2007) and March et al. (2011)—who claimed a coupling of torsion to *orbital* angular momentum, in particular in the context of the Gravity Probe B experiment—do not yield any information on torsion.

1. Introduction

Ever since E. Cartan in the 1920s enriched the geometric framework of general relativity (GR) by introducing a *torsion* of spacetime, the question arose whether one could find a measurement technique for detecting the presence of a torsion field. Mao et al. [1] claimed that the rotating quartz balls in the gyroscopes of the Gravity Probe B experiment [2], falling freely on an orbit around the Earth, should "feel" the torsion. Similarly, March et al. [3] argue with the precession of the Moon and the Mercury and extend later their considerations to the Lageos satellite.

A consistent theory of gravity with torsion emerged during the early 1960s as gauge theory of the Poincaré group, see the review in [4]. This Poincaré gauge theory of gravity incorporates as simplest viable cases the Einstein–Cartan(–Sciama–Kibble) theory (EC), the teleparallel equivalent $\mathrm{GR}_{||}$ of GR, and GR itself. So far, PG and, in particular, the existence of torsion have *not* been experimentally confirmed. However, PG is to be considered as the standard theory of gravity with torsion because of its very convincing gauge structure.

Since the early 1970s up to today, different groups have shown more or less independently that torsion couples only to the *elementary particle spin* and under no circumstances to the orbital angular momentum of test particles. This is established knowledge and we reconfirm this conclusion by discussing the energy–momentum law of PG, which has same form for all versions of PG. Therefore, we conclude that, unfortunately, the investigations of Mao et al. and March et al. do not yield any information on torsion.

2. Torsion defined, spin of matter introduced

Einstein's theory of gravitation, general relativity (GR), was finally formulated in 1916. Already since this time, mathematicians and physicists, namely Hessenberg, Levi-Civita, Weyl, Schouten, and Eddington, amongst others, started to develop the geometrical concept of a (linear) *connection* Γ. This is a tool for the parallel displacement of vectors in a differential manifold, in particular in 4-dimensional spacetime. The final formulation of the connection was given by E. Cartan in 1923/24. He defined the connection 1-form $\Gamma_\alpha{}^\beta = \Gamma_{i\alpha}{}^\beta dx^i$ as a new fundamental geometrical object (with $\alpha, \beta, \ldots$ as frame and $i, j, \ldots$ as coordinate indices, both running from 0 to 3); for the explicit references and for the formalism, including the conventions, compare [4], pp. 17–21.

If the connection is expressed purely in coordinate components, then the antisymmetric part of it is a tensor, Cartan's *torsion* tensor,

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$$T_{ij}{}^k = \Gamma_{ij}{}^k - \Gamma_{ji}{}^k \equiv 2\Gamma_{[ij]}{}^k = -T_{ji}{}^k, \tag{1}$$

with its 24 independent components. This is the tensor alluded to in the title of our Letter. Mao et al. [1] wanted to sense torsion by using the results of the Gravity Probe B experiment of Everitt et al. [2]; later, March et al. [3] tried to do the same thing by using data of the Moon, of the Mercury, and of the Lageos satellite. We will come back to this issue later.

In GR, the Riemannian connection is represented by the Christoffel symbols $\widetilde{\Gamma}_{ij}{}^k := \frac{1}{2} g^{kl}(\partial_i g_{jl} + \partial_j g_{li} - \partial_l g_{ij})$, where g_{ij} are the components of the metric tensor and $\partial_i := \partial/\partial x^i$. The Riemannian connection is symmetric, it is torsion-free, that is, $\widetilde{T}_{ij}{}^k = 0$. Massive test particles in GR move along the geodesics of the Riemannian connection:

$$\frac{d^2x^k}{d\tau^2} + \widetilde{\Gamma}_{ij}{}^k \frac{dx^i}{d\tau}\frac{dx^j}{d\tau} = 0. \tag{2}$$

When Cartan extended the geometrical framework of GR by introducing a torsion of spacetime, he was conscious of the fact that he also had to use a more fine-grained description of matter than in GR. Instead of a classical fluid, acting via a symmetric energy–momentum density $\mathfrak{t}$, he suggested a Cosserat type fluid with an asymmetric energy–momentum density $\mathfrak{T}$ and an intrinsic or spin angular momentum density $\mathfrak{S}$, see [4], pages 21 and 103.

This conception has been developed even before the spin of the electron was discovered. We recognize that the introduction of the geometrical concept of a torsion goes hand in hand with ascribing to matter, besides an energy–momentum density, a further dynamical characteristics, namely a spin angular momentum density. In a general-relativistic theory of gravity, torsion and spin are interdependent.

This interdependence was clear to Cartan. However, because of an unfounded assumption, see Section 7, he was not able to formulate a consistent theory of gravity with torsion.

3. Poincaré gauge theory as standard torsion theory

In the early 1960s, a consistent framework for a valid physical theory of torsion was initiated by Sciama [5] and Kibble [6]. It was conceived as a gauge theory of the Poincaré group [6], the semi-direct product of the *translations* (4 parameters) and the *Lorentz rotations* (6 parameters). In Minkowski spacetime, the Poincaré group acts rigidly ("globally"). By means of the gauge procedure à la Weyl–Yang–Mills, the Poincaré group is "localized", acts merely locally. This is made possible by introducing 4 gauge potentials for the translations and 6 gauge potentials for the Lorentz rotations. The emerging theory is called *Poincaré gauge theory* of gravitation (PG), see [4], Part B for details.

The arena of the PG is a *Riemann–Cartan* (RC) *spacetime*. It is determined by a metric $g_{\alpha\beta}$ (and its reciprocal $g^{\gamma\delta}$), an orthonormal coframe $\vartheta^\alpha = e_i{}^\alpha\, dx^i$, and a Lorentz connection $\Gamma^{\alpha\beta} := g^{\alpha\gamma}\Gamma_\gamma{}^\beta = -\Gamma^{\beta\alpha} = \Gamma_i{}^{\alpha\beta}\, dx^i$. Having such a connection, we can define a covariant exterior derivative D. For a RC-space, we find $Dg_{\alpha\beta} = 0$ (vanishing nonmetricity).

The coframe ϑ^α can be understood as translational gauge potential and the Lorentz connection $\Gamma^{\alpha\beta} = -\Gamma^{\beta\alpha}$ as rotational gauge potential. The corresponding gravitational field strengths are torsion and curvature, respectively, which we find by differentiation of the corresponding potentials:

$$T^\alpha := D\vartheta^\alpha = d\vartheta^\alpha + \Gamma_\beta{}^\alpha \wedge \vartheta^\beta, \tag{3}$$

$$R^{\alpha\beta} := d\Gamma^{\alpha\beta} - \Gamma^{\alpha\gamma} \wedge \Gamma_\gamma{}^\beta = -R^{\beta\alpha}. \tag{4}$$

Note that in the term $\Gamma_\beta{}^\alpha \wedge \vartheta^\beta$ of (3) the rotations and translation mix algebraically, due to the semi-direct product structure. Hence it has to be taken with a grain of salt that T^α is called

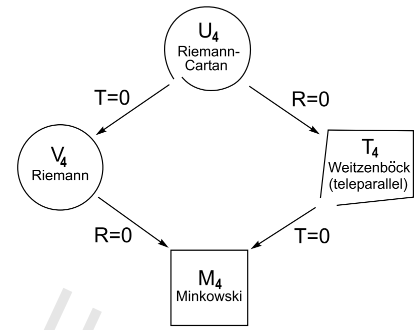

Fig. 1. A Riemann–Cartan space U_4 with torsion T and curvature R and its different limits (nonmetricity vanishes: $Q_{\alpha\beta} := -Dg_{\alpha\beta} = 0$), see [4], p. 174.

the translation field strength. In (4), the second term on the right-hand side $-\Gamma^{\alpha\gamma} \wedge \Gamma_\gamma{}^\beta$ is due to the non-commutative structure of the Lorentz rotations: they form a non-Abelian sub-algebra of the Poincaré algebra.

The different limits of a RC space are represented in Fig. 1. GR takes place in a V_4, PG in a U_4, $\mathrm{GR}_{||}$ in a T_4, and, when gravity can be neglected, we are in an M_4.

The definition (3) of the torsion, written with respect to coordinates, degenerates to (1). Moreover, the explicit form of the Lorentz connection, spelled out in coordinate indices, is $\Gamma_{ij}{}^k = \widetilde{\Gamma}_{ij}{}^k - K_{ij}{}^k$, with the contortion tensor

$$K_{ij}{}^k = -\frac{1}{2}\left(T_{ij}{}^k - T_j{}^k{}_i + T^k{}_{ij}\right) = -K_i{}^k{}_j. \tag{5}$$

So much about the geometry of the PG.

The physics of the PG is determined by a Lagrange 4-form

$$L = V\left(g_{\alpha\beta}, \vartheta^\alpha, T^\alpha, R^{\alpha\beta}\right) + L_{\mathrm{mat}}\left(g_{\alpha\beta}, \vartheta^\alpha, \Psi, D\Psi\right). \tag{6}$$

V is the gravitational gauge part of the Lagrangian, depending on the geometrical field variables, L_{mat} is the matter Lagrangian depending on some *minimally coupled* matter fields $\Psi(x)$, a Dirac field, for example. For special considerations referring to nonminimal coupling, compare Section 8.

By varying with respect to the gauge potentials (δ denotes a variation), we can read off the sources in the field equations of the PG as

$$\mathfrak{T}_\alpha = \frac{\delta L_{\mathrm{mat}}}{\delta\vartheta^\alpha} \quad \text{and} \quad \mathfrak{S}_{\alpha\beta} = \frac{\delta L_{\mathrm{mat}}}{\delta\Gamma^{\alpha\beta}} = -\mathfrak{S}_{\beta\alpha}, \tag{7}$$

respectively. They turn out to be the canonical 3-forms of *energy–momentum* $\mathfrak{T}_\alpha$ and of *spin angular momentum* $\mathfrak{S}_{\alpha\beta}$ of matter.[1]

We postpone the discussion of the explicit form of the gravitational Lagrangian V since this is not necessary for the understanding of the equations of motion of test particles in PG. We will only use it later in order to see that PG embodies viable gravitational theories, namely GR, Einstein–Cartan theory, and the teleparallel equivalent of GR.

4. How does one measure torsion of spacetime?

We have now a general idea how a PG looks like. We recognize that PG is a straightforward extension of GR, and we wonder, how a test particle moves in a spacetime with torsion.

Clearly, we will take recourse to the established methods of GR. GR is the only theory of nature in which the motion of a test

[1] They translate into the corresponding quantities of *tensor analysis* as follows: $\mathfrak{T}_\alpha = \mathcal{T}_\alpha{}^\beta \epsilon_\beta$ and $\mathfrak{S}_{\alpha\beta} = \mathcal{S}_{\alpha\beta}{}^\gamma \epsilon_\gamma$, with the 3-form density $\epsilon_a := e_\alpha \rfloor \epsilon$, the frame e_α, and the volume 4-form density ϵ. In the reverse order, we have $\vartheta^\beta \wedge \mathfrak{T}_\alpha = \epsilon\, \mathcal{T}_\alpha{}^\beta$ and $\vartheta^\gamma \wedge \mathfrak{S}_{\alpha\beta} = \epsilon\, \mathcal{S}_{\alpha\beta}{}^\gamma$.