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Tunneling of an energy eigenstate through a parabolic barrier viewed from Wigner phase space

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ABSTRACT

We analyze the tunneling of a particle through a repulsive potential resulting from an inverted harmonic oscillator in the quantum mechanical phase space described by the Wigner function. In particular, we solve the partial differential equations in phase space determining the Wigner function of an energy eigenstate of the inverted oscillator. The reflection or transmission coefficients R or T are then given by the total weight of all classical phase-space trajectories corresponding to energies below, or above the top of the barrier given by the Wigner function.

1. Introduction

Tunneling [1] of a particle through a barrier is one of the striking phenomena of quantum mechanics [2]. In the special case of a repulsive quadratic potential, corresponding for example to an inverted harmonic oscillator [3] shown in Fig. 1(a), the transmission coefficient T takes the form [4]

$$T = \frac{1}{1 + \mathrm{e}^{-2\pi\varepsilon}}, \tag{1}$$

depicted in Fig. 1(b). Here $\varepsilon \equiv E/(\hbar\Omega)$ is the scaled energy which is the ratio of the eigenvalue E and the natural energy parameter $\hbar\Omega$, where Ω is the steepness of the quadratic barrier and $\hbar$ denotes the Planck constant divided by 2π.

The expression Eq. (1) has played a crucial role in the context of nuclear fission [5]. It usually emerges [5] from a semiclassical analysis [6,7] of the Schrödinger equation of the inverted harmonic oscillator [3]. However, in the present Letter we rederive Eq. (1) from quantum phase space using the Wigner distribution function [8]. In particular, we show that Eq. (1) corresponds to the quantum mechanical weight of all *classical* trajectories [9] that have sufficient energy to go above the barrier.

This result is counterintuitive since in the standard formulation [2] of quantum mechanics à la Heisenberg and Schrödinger an energy eigenstate does not contain energies other than the eigenvalue. In contrast, the Wigner function [8] of such a state relies on the trajectories of all energies, however with positive or negative weights. Hence, the goal is to find these weights.

For this purpose we recall that two partial differential equations in phase space [10,11] determine the Wigner function of an energy eigenstate. They result from the commutator and the anti-commutator of the density operator with the Hamiltonian. The commutator yields the propagation equation of the Wigner function, that is, the quantum Liouville equation. In contrast, the anti-commutator leads to the phase-space analog of the Schrödinger eigenvalue equation.

For the case of an inverted harmonic oscillator the quantum Liouville equation reduces to the classical Liouville equation and is therefore, independent of $\hbar$. In particular, it shows that the Wigner function of an energy eigenstate is constant along the classical trajectories. However, even for a quadratic barrier, the equation following from the anti-commutator contains $\hbar$ explicitly. It is this equation which determines the quantum mechanical weight of each classical trajectory and provides us in this way with the tunneling and reflection coefficient.

This discussion also brings out most clearly the difference between a dynamical situation where a wave packet approaches a quadratic barrier and a stationary one corresponding to an energy eigenstate which is the topic of our Letter. Indeed, in the dynamical case, it is sufficient to propagate the Wigner function representing the initial wave packet along the classical trajectories as dictated by the reduction of the quantum Liouville equation to the classical. In this scenario $\hbar$ enters only through the initial state.

However, for the analysis of the energy eigenstate the propagation equation does not suffice. We also need to invoke the phase-space analog of the Schrödinger eigenvalue equation.

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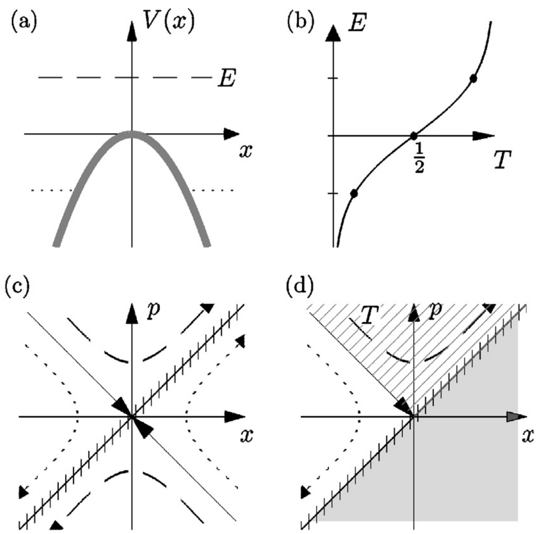

Fig. 1. Tunneling coefficient T of an energy eigenstate of eigenvalue E through a parabolic barrier (a) in its dependence on E (b) explained in terms of classical phase-space trajectories (c) subjected to the boundary conditions of a particle coming from the left (d). For three different energies — below, at the top of, and above the barrier — we depict the classical phase-space trajectories (c) which are either being reflected from, stopping at the top of, or going above the potential hill, respectively. The crossed line represents the separatrix in phase space separating the trajectories coming from the left and from the right. Hence, under normal scattering situations only half of phase space is accessible depicted in (d) for a particle approaching from the left. The quantum mechanical transmission curve (b) is due to the quantum mechanical weight of all classical trajectories going above the barrier provided by the Wigner function.

We emphasize that the Wigner function of tunneling in the inverted harmonic oscillator has also been analyzed in Ref. [12]. The authors of this paper first derive the quadrature representation of the energy eigenfunctions and then perform the integral in the definition of the Wigner function. In contrast, we start from the two partial differential equations [10,11] determining the Wigner function from phase space. Therefore, we find the Wigner function without ever going through the wave function. This approach is not only direct but also yields immediately the proposed interpretation of the tunneling coefficient. Moreover, it also builds a bridge to the 'on-first-sight' completely unrelated field of particle creation at event horizons of black holes associated with logarithmic phase singularities. Indeed, we show that as a result of the phase-space analog of the Schrödinger eigenvalue equation the kernel of the Wigner function contains such a singularity as well.

2. Phase-space differential equations

We study the tunneling of a particle of mass M through a quadratic barrier of steepness Ω expressed by the Hamiltonian

$$H \equiv \frac{p^2}{2M} - \frac{1}{2}M\Omega^2 x^2. \tag{2}$$

Here x and p denote the position and the coordinate of the particle.

For this purpose we consider the Wigner function [8]

$$W_E(x,p) \equiv \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty} \mathrm{d}y\, \mathrm{e}^{-\mathrm{i}py/\hbar}\, \psi_E^*\left(x-\frac{y}{2}\right)\psi_E\left(x+\frac{y}{2}\right) \tag{3}$$

of an energy eigenstate $|E\rangle$ of $\hat{H}$ with wave function $\psi_E = \psi_E(x)$. However, instead of solving first the time independent Schrödinger equation $\hat{H}\psi_E = E\psi_E$ for ψ_E and then performing the integration in Eq. (3) pursued in Ref. [12], we analyze the partial differential equations [10,11]

$$\left[\frac{p}{M}\frac{\partial}{\partial x} + M\Omega^2 x\frac{\partial}{\partial p}\right]W_E(x,p) = 0 \tag{4}$$

and

$$\left\{\left[\frac{p^2}{2M} - \frac{1}{2}M\Omega^2x^2\right] - \frac{\hbar^2}{8}\left[\frac{1}{M}\frac{\partial^2}{\partial x^2} - M\Omega^2\frac{\partial^2}{\partial p^2}\right]\right\} \times W_E(x,p) = EW_E(x,p) \tag{5}$$

for the Wigner function in phase space. We emphasize that Eqs. (4) and (5) are exact for the inverted harmonic oscillator.

3. Wigner function

The classical Liouville equation (4) implies that W_E is constant along the classical phase-space trajectories of a fixed energy H given by Eq. (2) and shown in Fig. 1(c), that is

$$W_E(x,p) = \mathcal{W}_{E/(\hbar\Omega)}\left(\frac{H(x,p)}{\hbar\Omega}\right). \tag{6}$$

Next we take into account the boundary conditions associated with a scattering process. Two distinct possibilities offer themselves: (i) the particle approaches the barrier from the left, or (ii) it impinges from the right.

The two cases manifest themselves in different classical phase-space trajectories. Whereas the situation (i) is described by the trajectories in the domain above the separatrix

$$p = M\Omega x, \tag{7}$$

depicted in Fig. 1(d), the case (ii) covers the area below it.

Hence, for a particle coming from the left, the Wigner function $W_E^{(l)}$ of an energy eigenstate reads

$$W_E^{(l)}(x,p) = \mathcal{W}_{E/(\hbar\Omega)}\left(\frac{H(x,p)}{\hbar\Omega}\right)\Theta(p - M\Omega x), \tag{8a}$$

where Θ denotes the Heaviside step function. Hence, only the classical trajectories above the separatrix contribute to the Wigner function as shown in Fig. 1(d).

Likewise, for a particle approaching from the right we find

$$W_E^{(r)}(x,p) = \mathcal{W}_{E/(\hbar\Omega)}\left(\frac{H(x,p)}{\hbar\Omega}\right)\Theta(M\Omega x - p). \tag{8b}$$

With the help of the familiar identity

$$x\delta(x) = 0 \tag{9}$$

for the Dirac delta function it is easy to verify that both expressions satisfy the Liouville equation (4) as long as the function $\mathcal{W}_\varepsilon$ is differentiable. The form of $\mathcal{W}_\varepsilon = \mathcal{W}_\varepsilon(\eta)$ corresponding to the scaled eigenvalue $\varepsilon \equiv E/(\hbar\Omega)$ in its dependence on the dimensional energy

$$\eta \equiv \frac{H(x,p)}{\hbar\Omega} \equiv \frac{1}{\hbar\Omega}\left[\frac{p^2}{2M} - \frac{1}{2}M\Omega^2x^2\right] \tag{10}$$

of a classical trajectory is then determined by the Schrödinger equation (5) in phase space. Indeed, when we substitute the ansatz Eq. (8b) into Eq. (5) we arrive at the ordinary differential equation

$$\eta\frac{\mathrm{d}^2\mathcal{W}_\varepsilon}{\mathrm{d}\eta^2} + \frac{\mathrm{d}\mathcal{W}_\varepsilon}{\mathrm{d}\eta} - 4(\varepsilon - \eta)\mathcal{W}_\varepsilon = 0. \tag{11}$$

Again we have made use of Eq. (9). It is remarkable that Eq. (11) is independent of the Heaviside step function.