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Minimum uncertainty states in angular momentum and angle variables for charged particles in structured electromagnetic fields

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ABSTRACT

We study the phase-space properties of a charged particle in a static electromagnetic field exhibiting vortex pairs with complementary topological charges and in a pure gauge field. A stationary solution of the Schrödinger equation that minimizes the uncertainty relations for angular momentum and trigonometric functions of the phase is obtained. It does not exhibit vortices and the angular momentum is due to the gauge field only. Increasing the topological charge of the vortices increases the regions where the Wigner function in the angle–angular momentum plane takes negative values, and thus enhances the quantum character of the dynamics.

1. Introduction

In recent years, it has been possible to create electric and magnetic fields in the laboratory with predetermined spatial topological defects. Thus, for instance, semiconductors in the presence of temperature gradients or external electric currents can develop steady current vortices and magnetic field vortices [1]. As for radiation electromagnetic (EM) fields, there has been much work on optical vortices: these are screw singularities in phase fronts characterized by a dark core and a topological charge determined by an azimuthal quantum number. The best known examples of beams carrying optical vortices are the Laguerre [2] and Bessel beams [3]. The transmission of these topological defects from the EM field to the quantum states of matter has been theoretically studied (see, e.g., Ref. [4]) and experimentally observed [5].

In quantum physics, the behavior of charged particles in EM fields with nontrivial topologies has many additional interesting properties, as it is well known since the pioneering paper of Aharonov and Bohm [6] (AB). In this case, a holonomy of the gauge field has physically observable effects.

The purpose of the present Letter is twofold. First, we study a class of quantum states of charged particles in a background field which is taken as the superposition of two fields with vortices of complementary topological charge, and an additional gauge field of AB type. Our primary aim is to illustrate the fact that even though an EM field exhibits a complicated topological structure, the quantum mechanical state of a charged particle in such a field may be relatively simple and without topological defects. Nevertheless, as shown in the second part of the Letter, the state under consideration is of minimum uncertainty in the angular momentum–angle observables [7], and the mean value of its angular momentum is proportional to the magnetic flux quantum associated to the gauge field.

The plan of the Letter is as follows. In Section 2, we present the static electric and magnetic background fields described above and their sources. In Section 3, we show that the Schrödinger equation for a charged particle in such a field configuration admits a normalized analytical solution, and we analyze its properties in detail. Section 4 is devoted to a careful formulation of the uncertainty relations for angle and angular momentum operators, and to the application of these relations to our quantum state, showing that it generalizes the minimum uncertainty states studied in Refs. [8,9]. The experimental characterization of the state of a quantum system is usually performed through the reconstruction of quasi-probability distribution functions [10], such as the Wigner function [11,12]. It is known, however, that some care must be taken in defining this distribution function for angle and angular momentum variables [7]. Accordingly, in Section 5, we apply Mukunda's formulation of the Wigner functions [13,15] to our minimum uncertainty wave-functions and we also calculate its marginal distributions. Finally, we discuss the consequences of the reported results in Section 6.

2. Static electric and magnetic fields from a superposition of vortex fields

Consider an electrostatic potential with a spatial dependence in circular cylinder coordinates given by

$$\mathcal{V}(\rho,\varphi,z) = \mathcal{V}^{(n)}(\rho,\varphi) + \mathcal{U}(z),$$
$$\mathcal{V}^{(n)}(\rho,\varphi) = V_0 J_n(\rho/\rho_0)\cos(n\varphi), \tag{1}$$

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where $J_n(x)$ is the Bessel function of order n, V_0 determines the intensity of the potential, ρ_0 the natural length scale of the system, and $\mathcal{U}(z)$ is a potential with no topological defects. $\mathcal{V}^{(n)}$ can be generated by a charge density distribution

$$\mathcal{D}^{(n)}(\rho,\varphi) = -\frac{V_0}{4\pi\rho_0^2} J_n(\rho/\rho_0)\cos(n\varphi). \tag{2}$$

Since near the origin $J_n(x) \sim x^n$, the potential $\mathcal{V}^{(n)}$ is a superposition of two scalar vortices of order n around the z-axis,

$$J_n(\rho/\rho_0)\cos(n\varphi) \sim (\rho/\rho_0)^n \left(e^{in\varphi} + e^{-in\varphi}\right), \quad \rho \ll \rho_0.$$

The corresponding electric field is given by

$$\mathbf{E}^{(n)}(\rho,\varphi) = \frac{V_0}{2\rho_0} \mathcal{R}e\big(J_{n+1}(\rho/\rho_0)e^{i(n+1)\varphi}\mathbf{e}_- - J_{n-1}(\rho/\rho_0)e^{i(n-1)\varphi}\mathbf{e}_+\big), \tag{3}$$

where $\mathcal{R}e$ denotes the real part, and $\mathbf{e}_\pm = \mathbf{e}_x \pm i\mathbf{e}_y$. The electric potential is illustrated in Fig. 1 and the field $\mathbf{E}$ in Fig. 2(a), where its complex topological structure can be visualized. Evidently, there are sources and sinks representing positive and negative charge distributions.

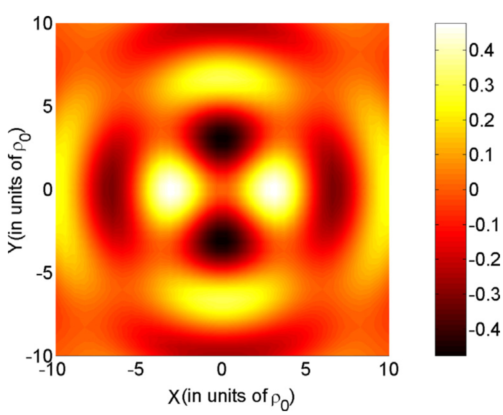

Fig. 1. Electric potential $\mathcal{V}^{(n)}(\rho,\varphi)$ for $n=2$.

In complete analogy, consider a magnetic field

$$\mathbf{B}(\rho,\varphi,z) = \mathbf{B}^{(n)}(\rho,\varphi) + B_s(\rho,\varphi,z)\mathbf{e}_z,$$
$$\mathbf{B}^{(n)}(\rho,\varphi) = B_0 J_n(\rho/\rho_0)\cos n\varphi\, \mathbf{e}_z, \tag{4}$$

where $B_s(\rho,\varphi,z)$ is the amplitude of a magnetic field without topological defects. Notice that in the configuration under consideration the electric $\mathbf{E}^{(n)}$ and magnetic $\mathbf{B}^{(n)}$ fields are orthogonal. The field $\mathbf{B}^{(n)}$ can be generated by a steady current

$$\mathbf{J}^{(n)}(\rho,\varphi) = -\frac{B_0}{8\pi c\rho_0} \mathcal{I}m\big(J_{n+1}(\rho/\rho_0)e^{i(n+1)\varphi}\mathbf{e}_- + J_{n-1}(\rho/\rho_0)e^{i(n-1)\varphi}\mathbf{e}_+\big), \tag{5}$$

and $\mathbf{B}$ results in general from a vector potential

$$\mathbf{A}(\rho,\varphi,z) = \mathbf{A}^{(n)}(\rho,\varphi) + \mathbf{A}_s(\rho,\varphi,z) + \nabla\phi(\rho,\varphi,z),$$
$$\mathbf{A}^{(n)}(\rho,\varphi) = -\frac{B_0\rho_0}{2} \mathcal{I}m\big(J_{n+1}(\rho/\rho_0)e^{i(n+1)\varphi}\mathbf{e}_- + J_{n-1}(\rho/\rho_0)e^{i(n-1)\varphi}\mathbf{e}_+\big),$$
$$\nabla^2\phi(\rho,\varphi,z) = 0. \tag{6}$$

In this equation, $\mathcal{I}m$ denotes the imaginary part, and $\nabla \times \mathbf{A}_s = B_z\mathbf{e}_z$. A function $\nabla\phi$ which does not give rise to a magnetic field $\mathbf{B}$ has been added: it is a pure gauge field. The field ϕ could be generated by a lengthy solenoid with arbitrarily small radius (line flux) lying on the z-axis. We work with $\mathbf{A}^{(n)}(\rho,\varphi)$ in Coulomb gauge $\nabla \cdot \mathbf{A}^{(n)} = 0$. In Fig. 2(b), we illustrate the structure of the field $\mathbf{A}^{(n)}$ which is proportional to $\mathbf{J}^{(n)}$. Current vortices are clearly visible.

3. Schrödinger equation

The electromagnetic field proposed in the previous section has been chosen with a view to obtaining a particular solution of the Schrödinger equation which is known to represent a minimum uncertainty state. Indeed, under stationary conditions, a spinless particle of charge q and mass m_0 satisfies the Schrödinger equation

$$\frac{1}{2m_0}\left[-i\hbar\nabla - \frac{q}{c}\mathbf{A}\right]^2 \Psi + q\mathcal{V}\Psi = E\Psi. \tag{7}$$

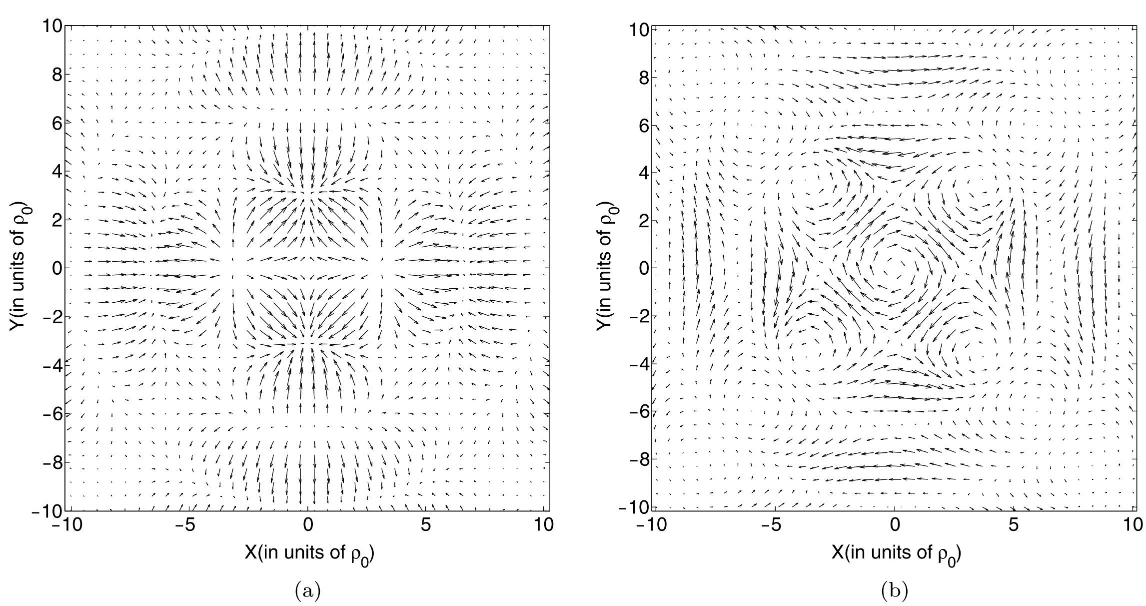

Fig. 2. Electric field $\mathbf{E}^{(n)}(\rho,\varphi)$ and magnetic potential $\mathbf{A}^{(n)}(\rho,\varphi)$ for $n=2$.