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Changes in the dynamics of two-dimensional maps by external forcing

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ABSTRACT

We investigate changes in periodicity, and even its suppression, by external periodic forcing in different two-dimensional maps, namely the Hénon map and the sine square map. By varying the amplitude of a periodic forcing with a fixed angular frequency, we show through numerical simulations in parameter-spaces that changes in periodicity may take place. We also show that windows of periodicity embedded in a chaotic region may be totally suppressed.

1. Introduction

A generic form of a forced two-dimensional discrete-time dynamical system can be written as the mapping [1]

$$x_{n+1} = f(x_n, y_n, \theta_n),$$
$$y_{n+1} = g(x_n, y_n),$$
$$\theta_{n+1} = \theta_n + \omega \quad (\text{mod } 1), \tag{1}$$

where f, g are real functions, x, y are the dynamical variables of the unforced system, $0 \leqslant \theta < 1$ is a discrete phase, and ω is an angular frequency. System (1) can be seen as a model for Poincaré maps of forced three-dimensional continuous-time systems. When the considered ω is rational, the forcing is said periodic, while for an irrational ω the forcing is said quasiperiodic. Nonlinear dynamical systems quasiperiodically forced, therefore considering an irrational ω in system (1), have been investigated by many authors [2–6], from the point of view of systems developing strange nonchaotic attractors. Such attractors are typical structures present in the phase-space, simultaneously displaying properties characteristic of both regular and chaotic behaviors, and were first described in Ref. [7].

The logistic map and the Hénon map, among others, have been used to illustrate the effect of quasiperiodic forcing on maps. For example, numerical and analytical investigations considering a quasiperiodically forced logistic map, show that the transition to strange nonchaotic attractors occurs when a period-doubled torus collides with its now unstable parent torus [2]. Therefore, the mechanism behind the development of strange nonchaotic attractors in a quasiperiodic forced logistic map is closely connected with the phenomenon of torus-doubling bifurcation. A scaling law for the torus-doubling bifurcation process was obtained numerically for a quasiperiodically forced logistic map [4]. This same important model has been considered to investigate the emergence of intermittent strange nonchaotic attractors in quasiperiodically forced period-doubling systems [5]. The Hénon map in turn, has been considered as a basic model to investigate the effect of a quasiperiodic forcing on two-dimensional invertible maps. The existence of strange nonchaotic attractors in these systems was verified [3], as well as the existence of intermittent strange nonchaotic attractors, the latter as a result of a collision between two torus, one being stable, the other being unstable [6].

The focus of the present Letter is to investigate numerically the effect of periodic forcing, instead of quasiperiodic forcing, in two two-dimensional maps, namely the Hénon map [8] and the sine square map [9,10]. In each case the angular frequency is kept fixed at different $\omega = \omega_0$, and parameter-space plots displaying the dynamical behavior are constructed, considering the parameters of the original unforced system, for some values of the amplitude ϵ. The goal is to investigate order–order and order–chaos transitions in these models, manifested, respectively, by changes in the periodicity and by the suppression of windows of periodicity embedded in chaotic regions, as the amplitude ϵ varies. In other words, this work deals with the manipulation of the nonlinear dynamics of two discrete-time system models, the invertible Hénon map and the non-invertible sine square map, by using an external periodic forcing. Our numerical results indicate that, under the influence of an external periodic forcing, change or even suppression of periodicity can be achieved in considerable regions of the parameter-space. The Letter is organized as follows. In Section 2

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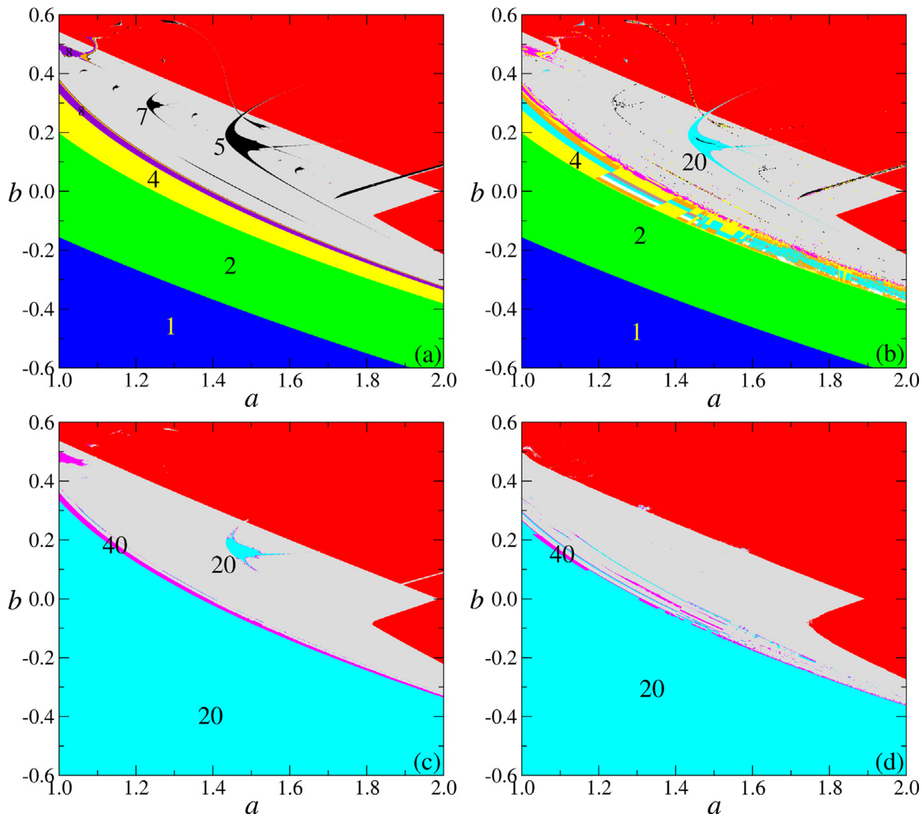

Fig. 1. Regions and colors in the (a, b) parameter-space of the forced Hénon map (2). (a) $\epsilon = 0$, therefore the unforced Hénon map. (b) $\epsilon = 10^{-5}$. (c) $\epsilon = 10^{-2}$. (d) $\epsilon = 10^{-1}$. (For interpretation of the references to color in this figure, the reader is referred to the web version of this Letter.)

we investigate the effect of a periodic forcing on the parameter-space of the Hénon map. Section 3 is dedicated to similar investigation considering the sine square map. To complete, concluding remarks are given in Section 4.

2. Periodically forced Hénon map

The periodically forced Hénon map here considered is given by

$$x_{n+1} = a - x_n^2 + b y_n + \epsilon \cos 2\pi\theta_n,$$
$$y_{n+1} = x_n,$$
$$\theta_{n+1} = \theta_n + \omega \pmod 1, \qquad (2)$$

where a and b are, respectively, the nonlinearity and the dissipation parameters of the original unforced Hénon map, and ϵ and ω, with ω rational, represent the amplitude and the angular frequency of the periodic forcing, respectively. As it happens with the unforced area-contracting Hénon map ($|b| < 1$), the periodically forced Hénon map is invertible, having also a nonzero constant Jacobian matrix determinant equal to $-b$, with $-1 < b < 1$. For all computations below, in this section, the ω value was kept fixed at 0.05, a rational number.

Fig. 1 shows four (a, b) parameter-space plots for the periodically forced Hénon map (2), for some values of the amplitude ϵ, namely $\epsilon = 0$ in panel (a), $\epsilon = 10^{-5}$ in (b), $\epsilon = 10^{-2}$ in (c), and $\epsilon = 10^{-1}$ in (d). Note that in fact, Fig. 1(a) is corresponding to the unforced Hénon map, because in this case the amplitude of the forcing is null. Each panel in Fig. 1 was obtained by discretizing the same parameter interval in a mesh of $10^3 \times 10^3$ points. For each point (a, b), an orbit from an arbitrary initial condition converges or to a chaotic attractor, or to a periodic attractor, or to an attractor located at infinity, after a transient of 50×10^3 iterations. Parameter values for which system (2) presents chaotic behavior, are corresponding to the region painted in grey color, while parameters for which system (2) converges to infinity are corresponding to the region painted in red color. Regions of parameter values related to some different periodicities are identified by integer numbers, which denote the period of the respective region.

In Fig. 1(a), that corresponds to the parameter-space of the unforced Hénon map, can be seen a piece of the known 1×2^n period-doubling bifurcation cascade, in the present scale specifically the more visible doublings 1 (blue region) $\Rightarrow$ 2 (green region) $\Rightarrow$ 4 (yellow region). Also can be seen some periodic structures embedded in the chaotic region, highlighting the typical shrimp-shaped period-5 in black color. As the external forcing is switched on and their amplitude is increased from zero, the picture in the parameter-space remains the same until $\epsilon = 10^{-6}$. For $\epsilon = 10^{-5}$, corresponding to the parameter-space in Fig. 1(b), the shrimp-shaped period-5 structure embedded in the chaotic region had their period changed to 20. When the forcing amplitude ϵ reaches the value 10^{-2}, the region in the parameter-space occupied by the piece of 1×2^n period-doubling bifurcation cascade is replaced by a large period-20 region, with a period-40 narrow strip. Simultaneously, the period-20 shrimp-shaped structure begins to disappear, as shown in Fig. 1(c). Fig. 1(d), where $\epsilon = 10^{-1}$, shows the chaotic region completely free of periodic structures, and also shows that the piece of the 1×2^n period-doubling bifurcation cascade abovementioned, was replaced by a single periodic region, namely the period-20 cyan region. Increasing more and more the forcing amplitude from $\epsilon = 0.1$, both the period-20 and the chaotic region begin to be swallowed by the divergence region. For $\epsilon = 2.1$, for example, the parameter-space appears entirely painted in red.

Therefore, we see above that by varying the amplitude of an external periodic forcing with a constant rational angular frequency, it is possible to destroy periodic windows embedded in a chaotic region of the (a, b) parameter-space of the Hénon map. It is possible to achieve continuous regions of chaos in the (a, b) parameter-space of the Hénon map, with no periodic windows embedded.