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Physics in space–time with scale-dependent metrics

Alexander S. Balankin

Grupo "Mecánica Fractal", ESIME-Zacatenco, Instituto Politécnico Nacional, México D.F., 07738, Mexico

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ABSTRACT

We construct three-dimensional space R^3_γ with the scale-dependent metric and the corresponding Minkowski space–time $M^4_{\gamma,\beta}$ with the scale-dependent fractal (D_H) and spectral (D_S) dimensions. The local derivatives based on scale-dependent metrics are defined and differential vector calculus in R^3_γ is developed. We state that $M^4_{\gamma,\beta}$ provides a unified phenomenological framework for dimensional flow observed in quite different models of quantum gravity. Nevertheless, the main attention is focused on the special case of flat space–time $M^4_{1/3,1}$ with the scale-dependent Cantor-dust-like distribution of admissible states, such that D_H increases from $D_H = 2$ on the scale $\ll \ell_0$ to $D_H = 4$ in the infrared limit $\gg \ell_0$, where ℓ_0 is the characteristic length (e.g. the Planck length, or characteristic size of multi-fractal features in heterogeneous medium), whereas $D_S \equiv 4$ in all scales. Possible applications of approach based on the scale-dependent metric to systems of different nature are briefly discussed.

1. Introduction

For a long time and even presently one of the most intriguing, and yet unsolved, problems in theoretical physics concerns the dimensionality of the space–time continuum [1–4]. Although the space in our world is commonly perceived as three dimensional and so the dimension of space–time continuum is $D = 3 + 1$, the problem of the actual dimension of our universe is still open. It is worth mentioning that the best experimental measurement of the dimensionality of our real world space is given by $d \approx 3 - 10^{-9}$ [5–7]. Moreover, the fractional dimensions of space–time emerge as useful concepts in several areas of physics [8–41]. Specifically, in the quantum field theories the dimension of space–time is a parameter that is commonly used to regularize divergent Feynman integrals in perturbative expansions [42]. Furthermore, fractional dimensional space represents an effective physical description of confinement in low-dimensional systems [10,11]. In this context it is also pertinent to note that fractal geometry dealing with systems of fractional dimensionality becomes a very useful tool in the material sciences, geology, and astrophysics (see Refs. [43–53] and the references therein).

Axiomatic basis for spaces of fractional dimension has been provided by Stillinger [8] along with a generalization of an integer dimensional Laplacian operator into a non-integer dimensional space S^D. Specifically, the fractional dimension D of space S^D is introduced by the fifth axiom which defines the integration rule in S^D according to which the integral depends on distances between the points along the integration path, but not on their absolute positions in S^D [8]. Accordingly, the Gaussian integral in S^D is defined as

$$\int_{S^D} dV_D \exp\left(ar^2\right) = (\pi/a)^{D/2}, \tag{1}$$

where a is a geometric constant and $r \in S^D$ is the radial coordinate [8]. Eq. (1) defines the fractional measure in S^D. In this respect, it is imperative to point out that although Stillinger [8] has not provided an explicit definition of metric in S^D, Eq. (1) implicitly implies the use of Euclidean metric $r = \sqrt{r_i r_j}$, where the Einstein summation convention is assumed. Furthermore, the space S^D was endowed with the generalized Laplacian Δ_α which in terms of mutually orthogonal coordinates $x_i \in S^D$ reads as

$$\Delta_\alpha = \partial_i \partial_j + (\alpha_{(i)} - 1)x_i^{-1}\partial_j, \tag{2}$$

where $\partial_i f$ denotes the conventional partial derivative with respect to x_i in E^n (no summation convention for indices in brackets), while $0 < \alpha_{(i)} \leqslant 1$ are dimensional exponents ($\sum_i^n \alpha_{(i)} = D < n$) [9]. The generalized Laplacian (2) was employed to formulate the Schrödinger wave mechanics and Gibbsian statistical mechanics in S^D [8,9] and used to develop the electromagnetic [53] and gravity theories in the fractional dimensional space [24]. It was also suggested that problems on fractals can be mapped into the corresponding problems in the fractional space [53].

Alternatively, to deal with problems on a fractal medium Φ^D_n within a continuum framework, Tarasov [16,17] has suggested the concept of fractal continuum. Generally, the fractal continuum can be defined as an n-dimensional region Φ^n_D of the embedding Euclidean space E^n filled with continuous matter (leaving no pores or

E-mail address: abalankin@ipn.mx.

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empty spaces) and equipped with appropriate fractional (or fractal) metric, measure, and rules of integration and differentiation, such that properties of Φ_D^n (density, displacements, velocities, etc.) are describable by the continuous (or, at worst, piecewise continuous) differentiable functions of space and time variables in E^n [50]. This definition implies that the density of admissible states characterizing how permitted places of particles are closely packed in $\Phi_D^n \subset E^n$ should be scale dependent to fulfill the definition of measure in a specific model [52]. Accordingly, it was suggested that problems on Φ_n^D with the fractal (Hausdorff, mass, box-counting, etc.) dimension D greater than its topological dimension $d^F < D$ can be mapped into the problems for fractal continuum of the same fractal dimension, whereas the topological dimension of $\Phi_D^n \subset E^n$ is per definition $d^{FC} = n > D$. In this way, Tarasov has adopted the left-sided Riemann–Liouville fractional integral to define the integration rule and employed the Riesz potential to define the fractal measure in $\Phi_D^3 \subset E^3$ with the Euclidean metric [52]. However, instead to use of the non-local or local fractional derivative allied with the left-sided Riemann–Liouville fractional integral, Tarasov has constructed the local differential operator expressed in terms of conventional derivatives which is also inverse to this integral and satisfies the general form of the Gauss–Green theorem for fractal manifolds [54]. In [19] the fractal continuum model was modified using the modified Riemann–Liouville fractional integral of Jumaire and the product measure, but the metric in $\Phi_D^3 \subset E^3$ was not defined. In [20,21] the fractal continuum model was constructed starting from the definition of fractal metric giving raise to corresponding fractional calculus and fractal measure in $\Phi_D^3 \subset E^3$. Different models of fractal continua were employed to study the mechanics, hydrodynamics, and electrodynamics of heterogeneous fractal media within a continuum framework [16–21,50,55].

On the other hand, the scale-dependent spectral dimension D_S of space–time has been proposed as a possible observable characterizing the geometry in discrete quantum gravity [56–65]. Specifically, it was suggested that the classical value $D_S = D = 4$ in the infrared (IR) is reduced to $D_S \leqslant 3$ in the ultraviolet (UV), whereas $D = 4$ in all scales. Accordingly, several models of quantum gravity are defined in the space–time of integer dimension $D = 3 + 1$, whereas the spectral dimension $D_S \leqslant D$ is scale dependent, but the value of D_S in the UV limit is different in different models (see Refs. [56–65]). Besides, authors of [66,67] have suggested the existence of scale-dependent metric associated with asymptotically safe scenarios in quantum Einstein gravity. More recently, Calcagni [68–70] has developed the field theory living in a multi-fractal space–time with scale-dependent D_H and D_S. In this model, the Minkowski space–time M_D^n was equipped with the multi-dimensional Lebesgue–Stieltjes measure $d\rho(x_i)$ whose form was obtained by arguments taken from fractal geometry using the rules of non-local integro-differential fractional calculus $Calc_\alpha = \{\partial^\alpha, I^\alpha, d^\alpha\}$, the fractional Laplacian Δ_α, and the 2α-norm

$$\|X\|_\alpha = \left|x_i^\alpha x_j^\alpha\right|^{1/2\alpha}, \tag{3}$$

where the range of admissible α is restricted by the interval $0.5 \leqslant \alpha \leqslant 1$ [70]. Consequently, the distance between two points $X, Y \in M_D^n$ in M_D^n is defined as $r_\alpha(X, Y) = \|X - Y\|_\alpha$. Furthermore, it was shown that the Hausdorff dimension of the Minkowski space–time M_D^n is $D_H = \beta + \alpha(n-1)$, while the spectral dimension is $D_S = \beta D_H$, where $\beta = 2/D_W$ and D_W is the dimension of random walk in M_D^n [68–70]. In this context, it is pertinent to point out that there are many other dynamic systems displaying scale dependence of fractal dimension (see, for example, [71–75]).

In this Letter, we suggest that physical phenomena in systems exhibiting dimension flow can be described in the Minkowski space–time endowed with appropriate scale-dependent metrics and the local differential vector calculus based on scale-dependent space metric.

2. Scale-dependent spatial metric and measure in *n*-dimensional space

First of all, let us construct n-dimensional space R_γ^n with the conventional Euclidean norm $\|\bar{r}\| = \sqrt{x_i x_j}$, but endowed with the scale-dependent metric, the corresponding weighted Lebesgue measure, and an appropriate local differential vector calculus. In this way, instead of the metric related to the Euclidean norm, here the distance between any pair of points $X, Y \in R_\gamma^n$ is defined as follows

$$r_\gamma(X, Y) = \sqrt{\delta_i \delta_j}, \tag{4}$$

where

$$\delta_i(x_i, y_i) = \chi^{(i)}(x_i)x_i - \chi^{(i)}(y_i)y_i, \quad \text{while } \chi^{(i)}(\ell_i) = 1 + (1-\gamma)(\ell_i/\ell_0)^{\gamma-1}, \tag{5}$$

ℓ_0 is the characteristic length (e.g. the Planck length $\ell_P \approx 1.7 \cdot 10^{-35}$ m, or characteristic size of multi-fractal features in heterogeneous medium), and $0 < \gamma \leqslant 1$. Physically, "weight" functions $\chi^{(i)}(\ell_i)$ account the scale-dependent distribution of admissible states between X and Y in R_γ^n, such that the density of states $\psi_{(i)}(x_i) = \lim_{y_i \to x_i}[\delta_i(x_i, y_i)/(x_i - y_i)]$ characterizes how the permitted places between two points are closely packed in R_γ^n [52]. It is a straightforward matter to verify that the distance defined by Eq. (4) together with Eq. (5) satisfies all conventional requirements for metric. That is: $r_\gamma \geqslant 0$, $r_\gamma(X, Y) = r_\gamma(Y, X)$, $r_\gamma(X, X) = 0$, and if $r_\gamma(X, Y) = 0$ then $X = Y$, while $r_\gamma(X, Y) + r_\gamma(X, Z) \geqslant r_\gamma(Y, Z)$. Notice also that in the limiting case of $\gamma = 1$ Eqs. (4), (5) define the Euclidean metric and so $R_{\gamma=1}^n$ is essentially the Euclidean space E^n. Furthermore, in space $R_{\gamma<1}^n$ the Euclidean metric is effectively restored at scales of $\ell_i \gg \ell_0$ ($i = 1, 3, \ldots, n$).

The weighted multi-dimensional Lebesgue measure allied with metric (4), (5) can be presented in the form

$$dV_{\gamma n}(x_i) = \prod_{i=1}^{n} \psi_{(i)}\, dx_i, \quad \text{where } \psi_{(i)} = 1 + \gamma(1-\gamma)\left(\frac{x_i}{\ell_0}\right)^{\gamma-1}, \tag{6}$$

from which follows that the metric (e.g. Hausdorff) dimension d_H of $R_{\gamma<1}^n$ is scale dependent (see Fig. 1(a)). Specifically, on the "IR scale" $\ell_i \gg \ell_0$ the metric dimension asymptotes to the spatial (topological) dimension of $R_{\gamma<1}^n$, that is $d_H = n$, whereas in the "UV limit" $\ell_i \ll \ell_0$ the metric dimension achieves the minimum value $d_H = \gamma n < n$.

Furthermore, the metric defined by Eqs. (4), (5) implies that the local partial derivative in R_γ^n can be defined as

$$\partial_i^\gamma f = \lim_{\substack{y_i \to x_i \\ x_j = const}} \left[\frac{f(x_i) - f(y_i)}{\delta_i(x_i, y_i)}\right] = \left[\partial_i\left(\chi^{(i)} x_i\right)\right]^{-1} \partial_i f = \psi_{(i)}^{-1} \partial_i f. \tag{7}$$

Notice that (7) differs from the fractional derivative defined as $f^{(\gamma)}(x_i) = \lim_{h \downarrow 0}[\Delta^\gamma f / h^\gamma]$, where $\Delta^\gamma f$ is expressed as a fractional Taylor's series of $f(x_i)$ [26,32]. Furthermore, we can define the vector local differential operators in R_γ^3 as follows

$$\vec{\nabla}^\gamma \varphi = \psi_{(i)}^{-1} \vec{e}_i \partial_j \varphi = Grad_\gamma\, \varphi, \tag{8}$$

$$Div_\gamma\, \vec{f} = \vec{\nabla}^\gamma \cdot \vec{f}, \qquad Rot_\gamma = \vec{\nabla}^\gamma \times \vec{f}, \tag{9}$$

$$\Delta_\gamma \varphi = \vec{\nabla}^\gamma \cdot \vec{\nabla}^\gamma \varphi = \psi_{(i)}^{-2}\left[\partial_i \partial_j \varphi - \gamma(1-\gamma)^2 (x_i/\ell_0)^{\gamma-2} \partial_j \varphi\right], \tag{10}$$