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Conservative generalized bifurcation diagrams

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ABSTRACT

Bifurcation cascades in conservative systems are shown to exhibit a *generalized* diagram, which contains all relevant informations regarding the location of periodic orbits (resonances), their width (island size), irrational tori and the infinite higher-order resonances, showing the intricate way they are born. Contraction rates for islands sizes, along period-doubling bifurcations, are estimated to be $\alpha_I \sim 3.9$. Results are demonstrated for the standard map and for the continuous Hénon–Heiles potential. The methods used here are very suitable to find periodic orbits in conservative systems, and to characterize the regular, mixed or chaotic dynamics as the nonlinear parameter is varied.

1. Introduction

Almost all physical systems in nature are so complex that long time predictions are nigh impossible. This is a characteristic of nonintegrable chaotic systems whose effects are visible in celestial mechanics, plasma physics, general relativity, quantum physics, communications problems, hard beats, social and stock market behaviors, weather forecast, among others. Thus, the precise description of the dynamics in nonintegrable systems is essential for the understanding of nature. One intrinsic and fundamental phenomenon of such nonintegrable systems, is that realistic stable orbits may vanish (or be born) when the nonlinear parameter varies. This can lead to a complicated behavior with a cascade of new orbits, which is satisfactory described by a bifurcation diagram (see [1] and references therein). One example is the period doubling bifurcation (PDB) cascade, well understood in one-dimensional dissipative discrete systems containing one parameter, where the intervals in the parameter, between successive PDBs, tend to a geometric progression with an universal ratio of $1/\delta = 1/4.66$, with δ being the Feigenbaum constant. For two-dimensional dissipative systems, bifurcation cascades manifest themselves inside periodic stable structures in the two parameter space, which appear to be generic for a large class of systems. Shrimp-like structures are one example, and appear in the dissipative Hénon [2,3] and ratchet [4] discrete systems, in continuous models [5], among many others. Such structures allow an analysis of geometric approximation ratios, and provide a very clear understanding of the dissipative dynamics. In contrast to dissipative systems, for conservative nonintegrable systems the description of bifurcations cascades is much more complicated. It is known [1,6], that by magnifying stable points in the phase space, a mixture of surrounding stable and unstable fixed points is found. This repeats itself for every stable fixed point, as explained by the Poincaré–Birkhoff theorem [7]. The intricate way higher order stable orbits bifurcate, the islands around them vary and are interconnected, and irrational tori behave as the nonlinear parameter of the system changes, is an interesting problem that still deserves to be deeply investigated. For 2 degrees of freedom area-preserving discrete systems, the intervals in the parameter between successive PDBs, tend to a geometric progression with a ratio of $1/\delta_H \sim 1/8.72$ [1]. Results have also been extended to higher-dimensional systems (please see [8–11] for more details).

This work uses convergence properties of the Finite Time Lyapunov Exponent (FTLE) to explore the dynamics of conservative systems in a mixed plot: initial condition (IC) *versus* the nonlinear parameter. The location of stable orbital points is easily found in such plot, and conservative bifurcation diagrams recognized to have a generalized form, containing infinite sub-diagrams with all rational/irrational tori from the periodic orbits (POs), independent of the period. This is a nice complete description, and extension, from an early work [12], done for another dynamical system. Results are remarkable and show the very complex, self-similar and generic bifurcation structure in conservative systems with only one parameter. They also suggest that contraction rates for the islands sizes, along PDBs, approach the constant $\alpha_I \sim 3.9$. A detailed

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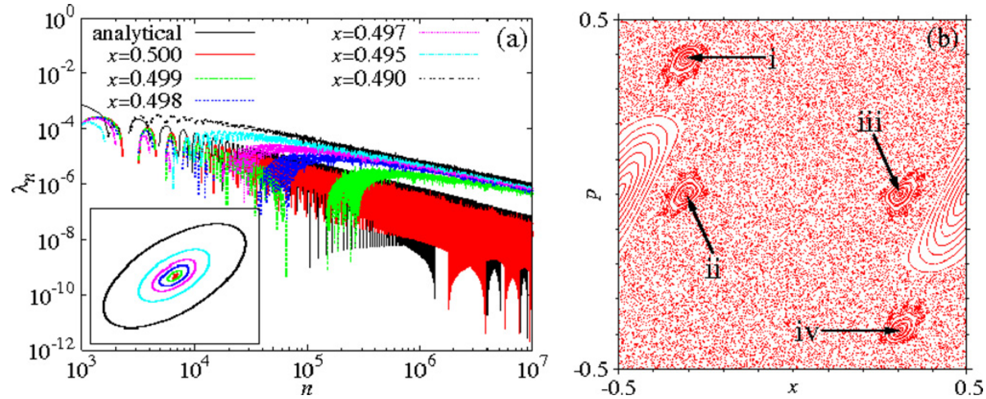

Fig. 1. (Color online.) (a) Log–log plot of the FTLEs (starting from below) as a function of the iteration time for the six trajectories shown in the inset (starting from the fixed point), and (b) phase space for $K = 2.6$.

numerical analysis is performed for the standard map, including the use of the first recurrence times instead the FTLEs, to demonstrate that results are independent of the method. A generalization is shown for the continuous Hénon–Heiles potential.

2. Discrete model: The standard map

To start it is appropriate to present results using a well known general model with wide applications, the Chirikov standard mapping, which is given by [13]:

$$\begin{cases} p_{n+1} = p_n + (K/2\pi)\sin(2\pi x_n) \mod 1, \\ x_{n+1} = x_n + p_{n+1} \mod 1. \end{cases} \tag{1}$$

K is the nonlinearity parameter and x_n, p_n are respectively position and momentum at discrete times $n = 1, 2, \ldots, N$. It is known that period-1 (shortly written per-1) fixed points are $p_1 = 1/2m$ (m integer) and $x_1 = 0, \pm 1/2$. The point $x_1 = 0$ is always unstable while $x_1 = \pm 1/2$ becomes unstable for $K > 4$. There exist also per-1 fixed points related to accelerator modes [13] whose stability condition is $|2 \pm K \cos x_{1l}| < 2$, with $K \sin x_{1l} = 2\pi l$ and l integer. For higher periods there are primary families of periodic points (which exist in the limit $K \to 0$) and bifurcation families which are born only for larger values of K (see [1] for more details).

3. Method: Finite time Lyapunov exponents

The key idea for the success of our proposal is the observation that the numerical convergence of the FTLEs is *distinct* for different ICs (same nonlinear parameter K), even between regular trajectories. To make this clear, consider, for example, that the IC is exactly on the stable (for $K \leqslant 4$) fixed point $x_0 = 1/2$. It can be calculated analytically that the corresponding Lyapunov exponent (LE), after one iteration, is exactly zero. But now consider that we start with an IC close to this fixed point, say $x'_0 = 1/2 + \Delta x_0$, which can be an irrational regular torus close to x_0. Assuming that $(\Delta x_0)^2 \approx 0.0$, the standard map can be linearized around x'_0 and the FTLE determined analytically, after n iterations, from the eigenvalues of the composed Jacobian $\{(1, -K), (1, 1-K)\}^n$. This FTLE is plotted in Fig. 1(a) as a black continuous line, and it converges exactly to zero only when $n \to \infty$. This means that ICs from stable tori around the fixed point, take a longer time to converge exactly to zero than the fixed point itself. This behavior can also be observed numerically by determining the FTLEs using Benettin's algorithm [14,15], which includes the Gram–Schmidt re-orthonormalization procedure. We use the six exemplary orbits shown in the inset of Fig. 1(a): the stable fixed point, demarked with a cross, and the five irrational tori around the fixed point. All trajectories are regular and have LEs exactly equal zero for infinite times. However, when calculating the FTLEs for the distinct ICs, we observe that ICs closer to the

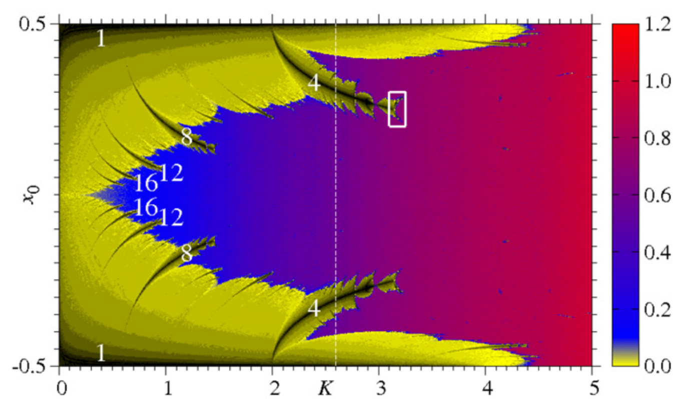

Fig. 2. (Color online.) FTLEs in the mixed space $x_0 \times K$ for the standard map with $p_0 = 0.0$, a grid of $10^3 \times 10^3$ points and 10^4 iterations.

fixed point converge faster to zero. In Fig. 1(a), the decay curves for the FTLEs *versus* times are plotted (starting from below) for the six ICs shown in the inset (starting from the fixed point). As ICs are taken more away from the periodic point, FTLEs converge slower to zero. The magnitudes of the FTLEs between different irrational tori are very small and not significant for any purpose. However, for the IC exactly on the PO, the FTLE converges faster to zero than other ICs around it. Essential to mention is that this is not a numerical convergence artefact due to the numerical method, but an analytical property, as shown above (continuous line in Fig. 1(a)). Thus, even though such small FTLEs are insignificant to distinguish between the irrational tori, they allow us to recognize where POs are *located* in phase space, and to understand the very complex and self-similar behaviors, which occur close to the POs as the nonlinear parameter K changes. For larger times, FTLEs in Fig. 1(a) continue to decrease linearly to zero, until the machine precision is reached, and when the actual method cannot be used anymore. Even though the method was explained using the standard map, it is equally applied to other conservative dynamical systems.

4. The generalized diagrams

Using the properties explained in Section 3, a very clarifying plot can be constructed, which allows us to recognize the bifurcation diagram in conservative systems in a simple way. Fig. 2 shows the FTLE (see colors bar) in the mixed space $x_0 \times K$ with $p_0 = 0.0$. Black lines are related to those ICs for which the FTLEs converge faster to zero and POs exist (this was checked explicitly for many black lines). Dark to light yellow points, around the main black lines, are related to irrational tori and also define the size of the corresponding island. These are regular trajectories for