Results in Physics 10 (2018) 272–276

Contents lists available at ScienceDirect

Results in Physics

journal homepage: www.elsevier.com/locate/rinp

Fractal calculus and its geometrical explanation

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ARTICLE INFO

Keywords:
Fractal temperature gradient
Hierarchical structure
Fractal derivative
Fractional derivative
Thermal resistance
Nanofiber membrane
Porous medium
Hausdorff derivative
Fractional differential equation

ABSTRACT

Fractal calculus is very simple but extremely effective to deal with phenomena in hierarchical or porous media. Its operation is almost same with that by the advanced calculus, making it much accessible to all non-mathematicians. This paper begins with the basic concept of fractal gradient of temperature, i.e., the temperature change between two points in a fractal medium, to reveal the basic properties of fractal calculus. The fractal velocity and fractal material derivative are then introduced to deduce laws for fluid mechanics and heat conduction in fractal space. Conservation of mass in a fractal space is geometrically explained, and an approximate transform of a fractal space on a smaller scale into its continuous partner on a larger scale is illustrated by a nanofiber membrane, which is smooth on any observable scales, but its air permeability has to studied in a nano scale, under such a small scale, the nanofiber membrane becomes a porous one. Finally an example is given to explain cocoon's heat-proof property, which cannot be unveiled by advanced calculus.

Introduction

Fractal geometry, fractal calculus and fractional calculus have been becoming hot topics in both mathematics and engineering for non-differential solutions. Fractal theory is the theoretical basis for the fractal spacetime [1,2], El Naschie's E-infinity theory [3], and life science [4] as well. Fractional calculus was introduced in Newton's time, and it has become a very hot topic in various fields, especially in mathematics and engineering for porous media [5–13], where classic mechanics becomes invalid to describe any phenomena on the porous size scale. For example, molecule diffusion in water is similar to a stochastic Brownian motion in view of continuum mechanics, but the diffusion follows fractal Fick laws if we observe the motion on a molecule scale. However, the fractional calculus is now such a mess that an engineer has no ability to select a suitable fractional derivative for his practical applications, most publications on fractional calculus are of pure mathematics though some authors claimed possible applications, and there are too many definitions on fractional derivative and new ones arise everyday [14–18]. Among all fractional derives, He's fractional derivative [19–21] and the local fractional derivative [22,23] are of mathematical correctness, physical foundation, and practical relevance. In 2012 the geometrical explanation of fractional calculus was given [24], and in 2014 a tutorial review was published on fractional calculus from its very beginning and physical understanding to practical applications [1].

Many researchers have already found the intrinsic relationship between the fractional dimensions and the fractional order [25]. This paper will focus itself on the fractal calculus, a relatively new branch of mathematics with easy understanding and ready applications.

Fractal calculus

The fractal calculus is relatively new, it can effectively deal with kinetics, which is always called as the fractal kinetics [26–28], where the fractal time replaces the continuous time. Nottale revealed that time does be discontinuous in microphysics [29], that means that fractal kinetics takes place on very small time scale.

The fractal derivative (Hausdorff derivative) on time fractal is defined as [30–36]

$$\frac{\partial T}{\partial t^{\sigma}} = \lim_{t_B \to x_A} \frac{T(t_B) - T(t_A)}{(t_B)^{\sigma} - (t_A)^{\sigma}} \tag{1}$$

where σ is the fractal dimensions of time.

A more general definition is given as follows [30–36]

$$\frac{\partial^{\tau} T}{\partial t^{\sigma}} = \lim_{t_B \to x_A} \frac{T^{\tau}(t_B) - T^{\tau}(t_A)}{(t_B)^{\sigma} - (t_A)^{\sigma}} \tag{2}$$

where τ is the fractal dimensions of space.

There are other definitions for fractal derivative, and we will not discuss all definitions, because some definitions are of only mathematical interest.

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https://doi.org/10.1016/j.rinp.2018.06.011
Received 18 May 2018; Received in revised form 4 June 2018; Accepted 5 June 2018
Available online 15 June 2018

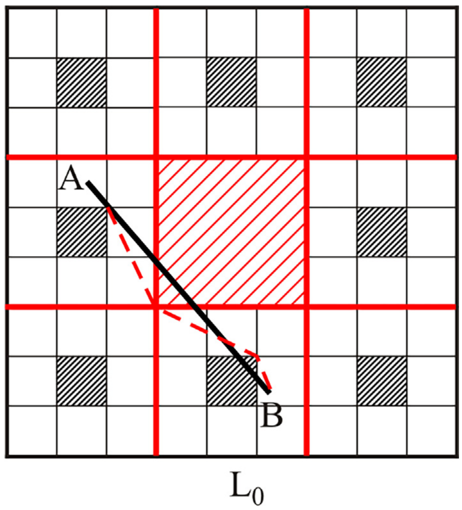

Fig. 1. Fractal gradient. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

Fractal gradient

To elucidate the basic ideas of the fractal calculus, we begin with the concept of gradient, which is widely used in mathematics and engineering. For the one-dimensional case, the gradient of temperature between two points A and B can be defined as

$$\frac{\Delta T}{\Delta x}=\frac{T_B-T_A}{x_B-x_A} \tag{3}$$

where T represents temperature or other variables. The gradient can be understood as the slope between two points:

$$\nabla T=\lim_{x_B\to x_A}\frac{T_B-T_A}{x_B-x_A} \tag{4}$$

For three-dimensional case, the gradient is defined as

$$\nabla T=\frac{\partial T}{\partial x}\mathbf{i}+\frac{\partial T}{\partial y}\mathbf{j}+\frac{\partial T}{\partial z}\mathbf{k} \tag{5}$$

The gradient is defined on a smooth space, and it becomes invalid for discontinuous space, and a new definition on a fractal space is much needed for practical applications.

In a fractal space as illustrated in Fig. 1, the gradient between points A and B cannot be described using the above definition.

We define average gradient, initial gradient, and terminal gradient, respectively, as follows

$$\overline{\nabla}T=\frac{\Delta T}{\Delta x}=\frac{T_B-T_A}{x_B-x_A} \tag{6}$$

$$\nabla_0 T=\lim_{x\to x_A}\frac{T-T_A}{x-x_A} \tag{7}$$

$$\nabla_\infty T=\lim_{x\to x_B}\frac{T_B-T}{x_B-x} \tag{8}$$

For a continuous space, we have

$$\overline{\nabla}T=\nabla_0 T=\nabla_\infty T \tag{9}$$

In a fractal space, however, the above equation becomes invalid, and we define a fractal gradient as follows

$$\nabla^\alpha T=\frac{T_B-T_A}{L_{AB}} \tag{10}$$

where L_{AB} is the length of the broken line in Fig. 1. According to fractal geometry, we have

$$L_{AB}=kL^\alpha \tag{11}$$

where L is distance between A and B, α is the fractal dimension value. In practical applications, hierarchical structure and porous medium can be approximately considered as a fractal space [1,7–10,24], that means there is a lowest hierarchy or minimal porous size. If the lowest hierarchical distance is L_0 (the side length of the shaded square in Fig. 1), beyond which no physical meaning exists. For example, L_0 is the nanoporous size of a nanofiber member [37–39], or the minimal porous size of a cocoon [40]. Using L_0, Eq. (9) can be updated as

$$L_{AB}=k_0(L_0)^\alpha \tag{12}$$

where k_0 is a constant.

When L_{AB} tends to extremely small but larger than L_0, we have [1,41]

$$L_{AB}=\frac{(L_0)^\alpha}{\Gamma(1+\alpha)} \tag{13}$$

We can define the fractal gradient in form [1,41]

$$\nabla^\alpha T=\Gamma(1+\alpha)\lim_{x_B-x_A\to L_0}\frac{T_B-T_A}{(x_B-x_A)^\alpha} \tag{14}$$

For the three dimensional case, the fractal gradient can be written in the form

$$\nabla^\alpha T=\frac{\partial T}{\partial x^\alpha}\mathbf{i}+\frac{\partial T}{\partial y^\alpha}\mathbf{j}+\frac{\partial T}{\partial z^\alpha}\mathbf{k} \tag{15}$$

where $\frac{\partial}{\partial x^\alpha}$ is the partial fractal derivative defined as [1,41]

$$\frac{\partial T}{\partial x^\alpha}=\Gamma(1+\alpha)\lim_{\Delta x=x_B-x_A\to L_0}\frac{T(x_B)-T(x_A)}{(x_B-x_A)^\alpha} \tag{16}$$

where α is the fractal dimensions in x-direction, L_0 is the lowest hierarchical distance.

The fractal derivative given in Eq. (16) has widely been used to deal with porous or hierarchical structures [42–45] with great success.

A fractal space is always not isotropic, that means the fractal dimensions in x-, y- and z-directions are different. We replace Eq. (16) by the following one

$$\nabla^{(\alpha,\beta,\gamma)}T=\frac{\partial T}{\partial x^\alpha}\mathbf{i}+\frac{\partial T}{\partial y^\beta}\mathbf{j}+\frac{\partial T}{\partial z^\gamma}\mathbf{k} \tag{17}$$

where α, β, and γ are, respectively, the fractal dimensions in x-, y- and z-directions,

$$\frac{\partial}{\partial x^\alpha}=\Gamma(1+\alpha)\lim_{x_B-x_A\to L_{0x}}\frac{T_B-T_A}{(x_B-x_A)^\alpha} \tag{18}$$

$$\frac{\partial}{\partial y^\beta}=\Gamma(1+\beta)\lim_{y_B-y_A\to L_{0y}}\frac{T_B-T_A}{(y_B-y_A)^\beta} \tag{19}$$

$$\frac{\partial}{\partial z^\gamma}=\Gamma(1+\gamma)\lim_{z_B-z_A\to L_{0z}}\frac{T_B-T_A}{(z_B-z_A)^\gamma} \tag{20}$$

where L_{0x}, L_{0y}, and L_{0z} are the minimal porous sized in x-, y- and z-directions, respectively.

One-dimensional heat equation with a source in a fractal medium can be written in the form

$$\frac{\partial T}{\partial t}+\frac{\partial}{\partial x^\alpha}\left(k\frac{\partial T}{\partial x^\alpha}\right)=Q_0 \tag{21}$$

where k is the material's conductivity, Q_0 is the heat source.

Three-dimensional heat equation with a source in a fractal medium reads

$$\frac{\partial T}{\partial t}+\frac{\partial}{\partial x^\alpha}\left(k_x\frac{\partial T}{\partial x^\alpha}\right)+\frac{\partial}{\partial y^\beta}\left(k_y\frac{\partial T}{\partial y^\beta}\right)+\frac{\partial}{\partial z^\gamma}\left(k_z\frac{\partial T}{\partial z^\gamma}\right)=Q_0 \tag{22}$$

where k_x, k_y, and k_z are, respectively, the material's conductivity in x-, y- and z-directions.

In order to establish laws in fractal media, it is necessary to introduce the concept of fractal velocity, which is defined as follows