



Exact traveling wave solutions of fractional order Boussinesq-like equations by applying Exp-function method



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ABSTRACT

We have computed new exact traveling wave solutions, including complex solutions of fractional order Boussinesq-Like equations, occurring in physical sciences and engineering, by applying Exp-function method. The method is blended with fractional complex transformation and modified Riemann-Liouville fractional order operator. Our obtained solutions are verified by substituting back into their corresponding equations. To the best of our knowledge, no other technique has been reported to cope with the said fractional order nonlinear problems combined with variety of exact solutions. Graphically, fractional order solution curves are shown to be strongly related to each other and most importantly, tend to fixate on their integer order solution curve. Our solutions comprise high frequencies and very small amplitude of the wave responses.

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Introduction

In recent years, nonlinear evolution equations (NLEEs) of fractional order have acquired a significant place in applied mathematics and engineering. These equations constitute remarkable applications in biomechanics, nonlinear optics, plasma physics, fluid dynamics, solid state physics and many other areas of physical sciences. Numerous methods have been proposed to look for exact solutions of NLEEs such as homogeneous balance method [1,2], Jacobi elliptic function method [3], Backlund transformations [4], functional variable method [5,6], tanh-function method [7–9], truncated painleve expansion method [10], (G/G) -expansion method [11,12], sine-cosine method [13], Hirota bilinear transformation method [14], F-expansion method [15,16], simple equation method [17] etc.

The Boussinesq-Like equations [18–21] are nonlinear evolution equations of the form

$$u_{tt} - u_{xx} - (6u^2u_x + u_{xxx})_x = 0, \quad (1)$$

$$u_{tt} - u_{xx} - (6u^2u_x + u_{xtt})_x = 0, \quad (2)$$

$$u_{tt} - u_{xt} - (6u^2u_x + u_{xxt})_x = 0, \quad (3)$$

and

$$u_{tt} - (6u^2u_x + u_{xxx})_x = 0, \quad (4)$$

where, u_{tt} and u_{xx} are second dissipative terms, u_{xxxx} is known as fourth spatial derivative and u_{xxxt} and u_{xttt} being mixed spatio-temporal derivative of the same order. The Eqs. (1)–(4) are linked to the good Boussinesq equation but then less conformable to the analytical techniques as these equations can no longer be fully integrable [19]. The spatio-temporal terms in (1)–(3) improve the properties of dispersion relation likewise the regularized Boussinesq equation. Above NLEEs appear in shallow water long waves, propagation of waves in elastic rods, coupled electrical circuits, vibration in nonlinear string, nonlinear lattice waves, dynamics of thin inviscid layers with free surface, and in the shape- memory alloys [13–16]. Solutions of Boussinesq-like equations are used for applying nonlinear water model to coastal and ocean engineering.

In this work, new exact traveling wave solutions of Boussinesq-Like equations (1)–(4) with their fractional order interpretations (related to the models discussed in [22,23] and several other papers):

$$D_{tt}^{2\alpha}u - D_{xx}^{2\beta}u - D_x^\beta(6u^2D_x^\beta u + D_{xxx}^{3\beta}u) = 0, \quad (5)$$

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$$D_{tt}^{2\alpha}u - D_{xx}^{2\beta}u - D_x^\beta(6u^2D_x^\beta u + D_{xtt}^{2\alpha+\beta}u) = 0, \quad 0 < \alpha, \beta \leq 1 \quad (6)$$

$$D_{tt}^{2\alpha}u - D_{xt}^{\alpha+\beta}u - D_x^\beta(6u^2D_x^\beta u + D_{xtt}^{\alpha+2\beta}u) = 0, \quad (7)$$

and

$$D_{tt}^{2\alpha}u - D_x^\beta(6u^2D_x^\beta u + D_{xxx}^{3\beta}u) = 0, \quad (8)$$

are obtained by applying Exp-function method [24,25] with the help of symbolic computation and is also applicable to differential equations of fractional order in a straightforward and simple way [26]. This method incorporates fractional complex transformation [27–29] and modified Riemann-Liouville operator [30,31] for seeking exact solutions of Boussinesq-Like physical models of fractional order. Among some other key techniques as mentioned earlier, the Exp-function method also gives exact solutions, and in most of the cases, a variety of solutions for a wide range of nonlinear problems of fractional as well as integer order. This makes the Exp-function method advantageous over the numerical, approximate analytical and semi numerical techniques. Further, some other studies related to solution methods can be seen in [36–59].

Theorem 1 [32]. Suppose that $u^{(r)}$ and $u^{(\gamma)}$ are respectively the highest order linear term and the highest order nonlinear term of a nonlinear ODE, where r and γ are both positive integers. Then the balancing procedure using the Exp-function ansatz

$$u(\xi) = \frac{\sum_{n=-c}^d a_n \exp(n\xi)}{\sum_{m=-p}^q b_m \exp(m\xi)}, \quad (9)$$

leads to $c = d$ and $p = q, \forall r, s, \Omega, \lambda \geq 1$.

Theorem 2 [32]. Suppose that $u^{(r)}$ and $u^{(s)}u^k$ are respectively the highest order linear term and the highest order nonlinear term of a nonlinear ODE, where r, s and Ω are all positive integers. Then the balancing procedure using the Exp-function ansatz leads to $c = d$ and $p = q, \forall r, s, k \geq 1$.

Theorem 3 [32]. Suppose that $u^{(r)}$ and $(u^{(s)})^\Omega$ are respectively the highest order linear term and the highest order nonlinear term of a nonlinear ODE, where r, s and Ω are all positive integers. Then the balancing procedure using the Exp-function ansatz leads to $c = d$ and $p = q, \forall r, s \geq 1, \forall \Omega \geq 2$.

Theorem 4 [32]. Suppose that $u^{(r)}$ and $(u^{(s)})^\Omega u^{(\lambda)}$ are respectively the highest order linear term and the highest order nonlinear term of a nonlinear ODE, where r, s, Ω and λ are all positive integers. Then the balancing procedure using the Exp-function ansatz leads to $c = d$ and $p = q, \forall r, s, \Omega, \lambda \geq 1$.

Modified Riemann-Liouville fractional derivative

To cope with the functions that are nondifferentiable, Jumarie presented new formulations for Taylor series of fractional sense and introduced modified Riemann-Liouville definition [30]. He made comparison with the Caputo-Djrbashian derivative [33,34], that defines a fractional operator of order less than 1 through a derivative which does not always work. Due to this reason, he presented modified Riemann-Liouville derivative [35]. Since then, modified Riemann-Liouville derivative has successfully been applied to a good deal of fractional order problems including our present work.

Jumarie's modified Riemann-Liouville operator [30,31] is defined as

$$D_t^\alpha f(t) = \begin{cases} \frac{1}{\Gamma(-\alpha)} \int_0^t (t-\tau)^{-\alpha-1} (f(\tau) - f(0)) d\tau, & \alpha < 0, \\ \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-\tau)^{-\alpha} (f(\tau) - f(0)) d\tau, & 0 < \alpha < 1, \\ [f^{(n)}(t)]^{\alpha-n}, & n \leq \alpha < n+1, n \geq 1. \end{cases} \quad (10)$$

Important properties of this particular operator are as follows

$$D_t^\alpha t^\gamma = \frac{\Gamma(1+\gamma)}{(1+\gamma-\alpha)} t^{\gamma-\alpha}, \quad \gamma > 0, \quad (11)$$

$$D_t^\alpha (cf(t)) = cD_t^\alpha f(t), \quad (12)$$

$$D_t^\alpha (af(t) + bgf(t)) = aD_t^\alpha f(t) + bD_t^\alpha g(t), \quad (13)$$

where, a, b and c are constants.

The Exp-function method

Consider the general nonlinear fractional partial differential equation containing higher order derivatives as well as nonlinear terms

$$P\left(u, \frac{\partial^\alpha u}{\partial t^\alpha}, \frac{\partial^\beta u}{\partial x^\beta}, \frac{\partial^\gamma u}{\partial y^\gamma}, \dots\right) = 0, \quad (14)$$

where u is a function to be known and P is a polynomial of u and its partial fractional order differential operators. By applying fractional complex transformation [27–29]

$$u = u(\xi), \quad \text{where } \xi = k \frac{x^\beta}{\Gamma(\beta+1)} + m \frac{y^\gamma}{\Gamma(\gamma+1)} - l \frac{t^\alpha}{\Gamma(\alpha+1)}, \quad (15)$$

we convert (4) into the following nonlinear ODE

$$Q(u, u', u'', u''', \dots) = 0 \quad (16)$$

In Exp-function method, we assume that the traveling wave solutions can be expressed in the form of (9) which can further be written as

$$u(\xi) = \frac{a_c \exp(c\xi) + \dots + a_{-d} \exp(-d\xi)}{b_p \exp(p\xi) + \dots + b_{-q} \exp(-q\xi)}. \quad (17)$$

(17) holds key role for finding analytic solution of given nonlinear problems. To attain the values of c and p , we balance the linear term of highest order in (16) with the highest order nonlinear term. In the same way, to obtain the values of d and q , we balance the linear term of lowest order in (16) with lowest order nonlinear term.

Applications of Exp-function method

Fractional order Boussinesq-like equations

(I) By using complex fractional transformation [27–29]

$$u = u(\xi), \quad \xi = k \frac{x^\beta}{\Gamma(\beta+1)} - l \frac{t^\alpha}{\Gamma(\alpha+1)}, \quad (18)$$

and modified Riemann-Liouville derivative, Boussinesq-like equation of the form (5) is converted into the ODE

$$l^2 u'' - (u'' + 12uu'^2 + 6u^2 u'')k^2 - u^{(4)}k^4 = 0, \quad (19)$$

that is integrated twice (neglecting constant of integration) to obtain

$$l^2 u - k^2(u + 2u^3) - k^4 u'' = 0. \quad (20)$$

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