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Stability and motion around equilibrium points in the rotating plane-symmetric potential field

Yu Jiang[a,b,*], Hexi Baoyin[b], Xianyu Wang[b], Hengnian Li[a]

[a] School of Aerospace Engineering, Tsinghua University, Beijing 100084, China
[b] State Key Laboratory of Astronautic Dynamics, Xi'an Satellite Control Center, Xi'an 710043, China

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ABSTRACT

This study presents a study of equilibrium points, periodic orbits, stabilities, and manifolds in a rotating plane-symmetric potential field. It has been found that the dynamical behaviour near equilibrium points is completely determined by the structure of the submanifolds and subspaces. The non-degenerate equilibrium points are classified into twelve cases. The necessary and sufficient conditions for linearly stable, non-resonant unstable and resonant equilibrium points are established. Furthermore, the results show that a resonant equilibrium point is a Hopf bifurcation point. In addition, if the rotating speed changes, two non-degenerate equilibria may collide and annihilate each other. The theory developed here is lastly applied to two particular cases, motions around a rotating, homogeneous cube and the asteroid 1620 Geographos. We found that the mutual annihilation of equilibrium points occurs as the rotating speed increases, and then the first surface shedding begins near the intersection point of the $-x$ axis and the surface. The results can be applied to planetary science, including the birth and evolution of the minor bodies in the Solar system, the rotational breakup and surface mass shedding of asteroids, etc.

Introduction

Space missions to minor bodies [1–5], such as asteroids, comets, and satellites around planets in the solar system, as well as the discovery of binary asteroids, make the dynamical behaviour in the vicinity of non-spherically shaped bodies (such as a massive inhomogeneous straight segment) a subject of increasing interest [6,7]. Some space missions consider flying a spacecraft around an asteroid and even landing on its surface [8], leading importance to the study of the dynamics in the potential field of an asteroid. In addition, the dynamics of a large-mass-ratio binary asteroid [9] that can be modelled as a massless particle flying around a large and irregularly shaped body (such as Ida and Dactyl, [10]) is also relevant to research concerning the motion near an irregularly shaped body.

The classical method of modelling celestial bodies is to expand the gravity potential using the Legendre polynomial series [11]; this method can provide a good approximation to nearly spherically shaped celestial bodies when the series is sufficiently long [12]. However, many minor bodies, such as asteroids, comets, and satellites around planets, have irregular shapes. For space missions to minor bodies, it is necessary to calculate the gravitational field of these irregular-shaped bodies. However, the method of the Legendre polynomial series does not converge at certain points [13,14] or regions [15]. Several methods are used to eliminate this difficulty.

Werner [16] developed a method that uses a polyhedron to model irregularly shaped bodies such as asteroids, comet nuclei, and small planetary satellites and then applied this method to calculate the gravitational field of the inner Martian satellite Phobos. Subsequently, the polyhedron method was applied to several asteroids, including asteroids 4769 Castalia [17], 4179 Toutatis [18], 216 Kleopatra [19–25] and the binary near-Earth asteroid (66391) 1999 KW4 [26–28].

However, the polyhedron model contains many free parameters and is highly complex; some simply shaped models may also yield good approximations for some bodies [13]. That is, although the polyhedron model offers higher precision for quantitatively analysing and computing the dynamical behaviour in the vicinity of some asteroids, the qualitative analysis of the dynamical behaviour in the vicinity of certain asteroids still may be achieved by considering simply shaped bodies. Thus, Elipe and Lara [13] have used a finite straight segment to study the equilibria, periodic-orbit families, bifurcations and stability regions in phase space in the vicinity of asteroid 433 Eros. Broucke and Elipe [29] have discussed the potential, periodic-orbit families and bifurcations in the potential field of a solid circular ring. Blesa [14] has presented several families of periodic orbits in the plane of a triangular

* Corresponding author at: School of Aerospace Engineering, Tsinghua University, Beijing 100084, China.
E-mail address: jiangyu_xian_china@163.com (Y. Jiang).

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plate and a square plate. Alberti and Vidal [30] have calculated the potential of a homogeneous annulus disk and have studied the orbital motion near the disk. Fukushima [31] has derived the acceleration of a uniform ring or disk. Liu et al. [32] have investigated the locations and linear stability of equilibria, periodic orbits around equilibria and heteroclinic orbits in the gravitational field of a rotating homogeneous cube. Li et al. [33] have investigated the locations and linear stability of equilibrium points as well as periodic orbits around equilibrium points in the vicinity of a rotating dumbbell-shaped body. These simply shaped bodies and potential fields, including the logarithmic gravity field [34], the straight segment [6,8,13,35], the solid circular ring [29,36], the triangular plate and the square plate [14], the homogeneous annulus disk [30,31], the homogeneous cube [32,37–39], the dumbbell-shaped body [33], the classical rotating dipole model [40–42], and the dipole segment model [43] are all plane-symmetric. The relative equilibria of spacecrafts in the second degree and order-gravity field [44,45] are different from the equilibria in the above studies. The second degree and order-gravity field is also a rotating plane-symmetric potential field. The relative equilibria of spacecrafts include the equilibria of the position and the attitude, and the position is also the relative equilibria in the plane-symmetric potential field.

In this work, we are interested in the study of the dynamics of orbits in a rotating plane-symmetric gravitational field (unless explicitly stated otherwise, all discussions concern the dynamics in this type of gravitational field), including equilibrium points, periodic orbits, and manifolds. The dynamical behaviours in the xy plane and the z axis are decoupled for the plane-symmetric case, and the topological classifications for the plane-symmetric case are different with the general cases. Thus we focus on the plane-symmetric case in the current study.

The linearised equations of motion relative to an equilibrium point are derived and investigated in Section “Equations of motion”. Furthermore, the characteristic equation of equilibrium points is presented. In Section “Periodic orbits and submanifolds near equilibrium points”, the structure of the submanifolds and subspaces near an equilibrium point are studied, which fixes the motion state near the equilibrium point. It is found that there are twelve cases for the non-degenerate equilibrium points in the plane-symmetric potential field of a rotating plane-symmetric body. The necessary and sufficient conditions for linearly stable, non-resonant unstable and resonant equilibrium points are presented. If the rotating speed varies, two non-degenerate equilibria may collide and change to one degenerate equilibrium point and then annihilate each other.

The theory developed in this study is then applied to the motion in the gravitational potential of a rotating homogeneous cube and asteroid 1620 Geographos [42] in Section “Applications”. In the gravitational potential of a rotating homogeneous cube, it is found that there are two families of periodic orbits on the xy plane near equilibrium points E1, E3, E5 and E7, and there is only one family of periodic orbits on the xy plane near the equilibrium points E2, E4, E6 and E8. While the rotation speed of asteroid 1620 Geographos varies, the number of equilibrium points will change from five to three to one. The positions of mutual annihilation of equilibrium points are inside the body of asteroid 1620 Geographos. The results can be applied to the scientific research of the birth and evolution of the Solar System and its minor bodies [46,47], the rotational breakup of asteroids and comets [48], as well as the surface mass shedding of asteroids [49] in the future Human mission to asteroids [48,50].

Equations of motion

Equations of motion in the arbitrary body-fixed frame

The potential field of a rotating plane-symmetric body satisfies

$$U(x, y, z) = U(x, y, -z), \tag{1}$$

where (x, y, z) is the coordinates in the body-fixed coordinate system, and U is the potential of the body.

Consider the motion of a massless particle in the potential field of a rotating plane-symmetric body; the dynamical system is a Hamiltonian system. The equations of motion of the particle relative to the body can be written as

$$\ddot{\mathbf{r}} + 2\boldsymbol{\omega}\times\dot{\mathbf{r}} + \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r}) + \dot{\boldsymbol{\omega}}\times\mathbf{r} + \frac{\partial U(\mathbf{r})}{\partial\mathbf{r}} = 0, \tag{2}$$

where $\mathbf{r}$ is the body-fixed vector from the centre of mass of the body to the particle, $\boldsymbol{\omega}$ is the rotational-angular-velocity vector of the body relative to the inertial frame of reference, and $\frac{\partial U(\mathbf{r})}{\partial\mathbf{r}}$ is the gradient of the potential. If $\boldsymbol{\omega} = 0$, then the body is fixed and has no rotation.

The Jacobian integral H is defined as

$$H = \frac{1}{2}\dot{\mathbf{r}}\cdot\dot{\mathbf{r}} - \frac{1}{2}(\boldsymbol{\omega}\times\mathbf{r})(\boldsymbol{\omega}\times\mathbf{r}) + U(\mathbf{r}). \tag{3}$$

H is time invariant if and only if $\boldsymbol{\omega}$ is time invariant. When H is time invariant, it is also called the Jacobian constant.

The effective potential can be defined as [20,51]

$$V(\mathbf{r}) = -\frac{1}{2}(\boldsymbol{\omega}\times\mathbf{r})(\boldsymbol{\omega}\times\mathbf{r}) + U(\mathbf{r}), \tag{4}$$

which satisfies

$$V(x, y, z) = V(x, y, -z). \tag{5}$$

For a uniformly rotating body, Eq. (2) can be simplified to

$$\ddot{\mathbf{r}} + 2\boldsymbol{\omega}\times\dot{\mathbf{r}} + \frac{\partial V(\mathbf{r})}{\partial\mathbf{r}} = 0. \tag{6}$$

The body is assumed to be uniformly rotating throughout this paper. The body-fixed frame can be defined via a set of orthonormal right-handed unit vectors $\mathbf{e}$:

$$\mathbf{e} \equiv \begin{Bmatrix} \mathbf{e}_x \\ \mathbf{e}_y \\ \mathbf{e}_z \end{Bmatrix}. \tag{7}$$

The frame of reference that is used throughout this study is the body-fixed frame. Let ω be the modulus of the vector $\boldsymbol{\omega}$; in addition, consider that the vector $\boldsymbol{\omega}$ can be written as $\boldsymbol{\omega} = \omega_x\mathbf{e}_x + \omega_y\mathbf{e}_y + \omega_z\mathbf{e}_z$. The equilibrium points are the critical points of the effective potential $V(\mathbf{r})$.

The linearised equations of motion relative to the equilibrium point can be written as

$$\begin{aligned} &\ddot{\xi} + 2\omega_y\dot{\zeta} - 2\omega_z\dot{\eta} + V_{xx}\xi + V_{xy}\eta = 0 \\ &\ddot{\eta} + 2\omega_z\dot{\xi} - 2\omega_x\dot{\zeta} + V_{yx}\xi + V_{yy}\eta = 0 \\ &\ddot{\zeta} + 2\omega_x\dot{\eta} - 2\omega_y\dot{\xi} + V_{zz}\zeta = 0. \end{aligned} \tag{8}$$

where $\xi = x - x_E$, $\eta = y - y_E$, $\zeta = z - z_E$. Here (x_E, y_E, z_E) represents the location of the equilibrium point.

The characteristic equation follows:

$$\begin{vmatrix} \lambda^2 + V_{xx} & -2\omega_z\lambda + V_{xy} & 2\omega_y\lambda \\ 2\omega_z\lambda + V_{yx} & \lambda^2 + V_{yy} & -2\omega_x\lambda \\ -2\omega_y\lambda & 2\omega_x\lambda & \lambda^2 + V_{zz} \end{vmatrix} = 0. \tag{9}$$

Furthermore, it can be rewritten as a sextic equation in λ:

$$\begin{aligned} &\lambda^6 + (V_{xx} + V_{yy} + V_{zz} + 4\omega_x^2 + 4\omega_y^2 + 4\omega_z^2)\lambda^4 + (V_{xx}V_{yy} + V_{yy}V_{zz} + V_{zz}V_{xx} \\ &\quad - V_{xy}^2 + 8\omega_x\omega_y V_{xy} + 4\omega_x^2 V_{xx} + 4\omega_y^2 V_{yy} + 4\omega_z^2 V_{zz})\lambda^2 \\ &\quad + (V_{xx}V_{yy}V_{zz} - V_{zz}V_{xy}^2) = 0, \end{aligned} \tag{10}$$

where λ denotes the eigenvalues of Eq. (8). The linear stability of the equilibrium point is determined by the six roots of Eq. (10). Let $\lambda_i (i = 1, 2, \ldots, 6)$ represent the roots of Eq. (10).