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Topological sensitivity based far-field detection of elastic inclusions ☆

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ABSTRACT

The aim of this article is to present and rigorously analyze topological sensitivity based algorithms for detection of diametrically small inclusions in an isotropic homogeneous elastic formation using single and multiple measurements of the far-field scattering amplitudes. A L^2-cost functional is considered and a location indicator is constructed from its topological derivative. The performance of the indicator is analyzed in terms of the topological sensitivity for location detection and stability with respect to measurement and medium noises. It is established that the location indicator does not guarantee inclusion detection and achieves only a low resolution when there is mode-conversion in an elastic formation. Accordingly, a weighted location indicator is designed to tackle the mode-conversion phenomenon. It is substantiated that the weighted function renders the location of an inclusion stably with resolution as per Rayleigh criterion.

Introduction

Inverse scattering has been of prime interest in recent decades due to a variety of applications in various branches of engineering and applied sciences [1–5]. The goal of these problems is to locate and characterize scatterers of different geometric nature such as inclusions, cracks, and cavities from the limited information of single or multiple scattered fields. Many promising computational and mathematical frameworks adaptive to different imaging and experimental setups have been developed to address these inverse problems over a span of last few decades (see, e.g., [6–16]). In particular, topological sensitivity frameworks have received significant attention for the reconstruction of location, shape or constitutive parameters of anomalies due to their simplicity and robustness (see, e.g., [17–33]).

In topological sensitivity frameworks, an inverse scattering problem is first converted to a minimization problem for a discrepancy functional by nucleating an anomaly at a search location in the background medium. The *topological sensitivity* of the misfit to nucleating an anomaly at different search locations, is determined by its topological derivative which serves as a location indicator or shape classifier.

Despite their extensive use, the quantitative analysis of the topological sensitivity inverse scattering frameworks for anomaly detection, in terms of resolution limit, signal-to-noise ratio and stability in the presence of medium or measurement noise, remains heuristic at large. First rigorous quantitative analysis for anti-plane elasticity was performed by Ammari et al. [34] using asymptotic expansions with respect to the size of inclusion. It

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was established that inclusion detection inside a bounded domain is only guaranteed if the measurements are pre-processed for the boundary effects and the inclusions are separated apart from the boundary. Towards this end, the *Calderón preconditioner* associated to the boundary of the domain was used as a filter. The performance of the framework was compared with a number of contemporary algorithms including Kirchhoff migration, back-propagation, and Multiple Signal Classification (MUSIC). The topological sensitivity algorithm with preconditioning appeared to be more stable and resolved than the listed algorithms but at a cost of increased computational complexity. It was observed in [35] that, even with the preconditioning, the localization using near-field measurements of the scattered fields was compromised in the general isotropic elasticity setting due to non-linear coupling and conversion of shear and pressure wave-modes. Accordingly, the classical framework was slightly modified using Helmholtz vector decomposition and assigning proper weights to the pressure and shear components in terms of respective wave speeds. In three-dimensional electromagnetic media in full Maxwell setting, the performance of topological sensitivity based inclusion detection functions with filtered boundary measurements of the tangential components of the magnetic fields was debated in [36]. The analysis was further extended in [37,38] to the detection of electromagnetic inclusions using single and multiple far-field scattering amplitudes.

The aim in this article is to design and debate the performance of the topological sensitivity algorithms for locating small inclusions of vanishing sizes in an unbounded elastic formation with single or multiple measurements of the elastic far-field patterns. The general case of a linear isotropic elastic medium is considered and a location indicator is constructed from a L^2-cost functional. A rigorous sensitivity analysis is performed based on the asymptotic expansions of the far-field scattering patterns versus the scale factor of the inclusion. It is established that the performance of the classical location indicator degenerates due to the mode-conversion (from longitudinal (P) to transverse (S) waveforms and vise-verse). Specifically, it is established that the indicator does not guarantee inclusion detection and achieves a low resolution than expected in the presence of mode-conversion. Accordingly, a weighted correction to the location indicator is proposed. Firstly, the pressure and the shear components of the back-propagator are decoupled using Helmholtz mode decomposition. Then, the modes of the back-propagator are correlated with the corresponding components of the incident fields. Finally, the resulting components are aggregated after assigning proper weights in terms of the corresponding wave speeds. In order to debate the capabilities of the weighted indicator, a rigorous sensitivity and stability analysis is performed when the measurements are corrupted by an additive noise or there is a random medium noise contaminating the far-field patterns.

It is worthwhile highlighting that the inverse elastic scattering problem caters to various applications including non-destructive evaluation of an elastic structure for integrity and material impurities [39], prospecting of mineral reservoirs [40], and medical diagnostics for detecting and classifying small tumors and locating tissue abnormalities of vanishing sizes [41,42].

The contents of this article are arranged in the following order. A mathematical description of the inverse problem is furnished in S ection "Mathematical formulation" along with a few preliminaries. The cost functional and the corresponding topological sensitivity based location indicators are introduced in Section "Location search using topological sensitivity indicators" and their topological sensitivity and resolution limits are analyzed. In Section "Statistical stability with measurement noise", the statistical stability of the proposed location indicators is discussed when the measurements are corrupted by an additive Gaussian noise. The statistical stability of the proposed indicators in a randomly fluctuating medium is debated in Section "Statistical stability with random medium noise". Finally, the main results of this article are summarized in Section "Conclusion".

Mathematical formulation

Let us mathematically introduce the inverse scattering problem undertaken in this article. The nomenclature of this investigation is provided in Section "Nomenclature and problem formulation" along with the mathematical description of the inverse problem dealt with. A few preliminaries are given in Section "Preliminaries " to facilitate the ensuing discussion. For details beyond those provided in this section, the readers are suggested to consult monograph [2].

Nomenclature and problem formulation

Let $\mathbb{R}^d$, $d = 2$ or 3, be loaded by an isotropic homogeneous elastic material (hereinafter referred to as the *background medium*) that has the volume density $\rho_0 \in \mathbb{R}_+$, and the Lamé parameters λ_0 (compressional modulus) and μ_0 (shear modulus) satisfying the strong convexity conditions,

$$\mu_0 > 0 \quad \text{and} \quad d\lambda_0 + 2\mu_0 > 0.$$

Let an isotropic and homogeneous elastic inclusion, represented by a bounded domain $D := \mathbf{z}_D + \epsilon B_D$ with $\mathcal{C}^2$ boundary ∂D, be embedded in the background medium. The position vector $\mathbf{z}_D \in \mathbb{R}^d$ is the center of mass of the inclusion D and the reference domain $B_D \subset \mathbb{R}^d$, assumed to be smooth and containing the origin, is the bulk of D. The scale factor $\epsilon \in \mathbb{R}_+$ determines the characteristic size of D. The inclusion is supposed to have the corresponding parameters $\rho_1 \in \mathbb{R}_+, \lambda_1$, and μ_1 which satisfy

$$\mu_1 > 0 \quad \text{and} \quad d\lambda_1 + 2\mu_1 > 0. \tag{1}$$

It is further assumed that

$$(\lambda_1 - \lambda_0)(\mu_1 - \mu_0) \geqslant 0, \tag{2}$$

which is required to ensure the positive (or negative)-definiteness of the associated EMT (see, Section "Elastic moment tensor"). Henceforth, the constitutive parameters of the medium in the presence of inclusion D are denoted by λ, μ, and ρ, i.e.,

$$(\lambda; \mu; \rho)(\mathbf{x}) := (\lambda_0; \mu_0; \rho_0)\chi_{\mathbb{R}^d \setminus \overline{D}}(\mathbf{x}) + (\lambda_1; \mu_1; \rho_1)\chi_D(\mathbf{x}),$$

where χ_D represents the characteristic function of domain D.

Let $\mathbf{w} : \mathbb{R}^d \to \mathbb{C}^d$ be a generic vector field and $\boldsymbol{\nu}(\mathbf{x}) : \partial D \to \mathbb{R}^d$ denote the outward unit normal at $\mathbf{x} \in \partial D$. Then, the linear elasticity operator and the surface traction on ∂D are, respectively, defined as

$$\mathcal{L}_{\lambda_0,\mu_0}[\mathbf{w}](\mathbf{x}) := \big(\lambda_0 \Delta \mathbf{w} + (\lambda_0 + \mu_0)\nabla\nabla \cdot \mathbf{w}\big)(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^d,$$
$$\frac{\partial \mathbf{w}}{\partial \nu}(\mathbf{x}) := \big(\lambda_0 (\nabla \cdot \mathbf{w})\boldsymbol{\nu} + 2\mu_0 \nabla^s \mathbf{w}\,\boldsymbol{\nu}\big)(\mathbf{x}), \quad \mathbf{x} \in \partial D.$$

The elasticity and surface traction operators associated with the parameters $(\lambda_1; \mu_1)$ are defined analogously and are denoted by $\mathcal{L}_{\lambda,\mu}$