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The effective Young's modulus of composites beyond the Voigt estimation due to the Poisson effect

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ABSTRACT

The Voigt estimation or the rule of mixture has been believed to be the upper bound of the effective Young's modulus of composites. However, this is only true in the situations where the Poisson effect is not significant. In this paper, we accurately derived the effective compliance matrix for two-phase layered composites by accounting for the Poisson effect. It is interesting to find that the effective Young's modulus in both transverse and longitudinal direction can exceed not only the Voigt estimation, but also the Young's modulus of the stiffest constituent phase. Moreover, the longitudinal (or parallel connection) Young's modulus is not always larger than the transverse (or serial connection) one. For isotropic composites, it has also been demonstrated that the Voigt estimation is not the upper bound for the effective Young's modulus. Therefore, one should be careful in applying the well known bound estimations on the effective Young's modulus of composites if one of the phases is near its incompressibility limit.

1. Introduction

The upper and lower bounds for the effective stiffness of two-phase composites have been studied for a long time. Among these works, Voigt [1] adopted isostrain assumption to obtain the estimation of the effective composite stiffness matrix as the weighted volume average of the stiffness matrices of constituent phases. Based on the isostress assumption, Reuss [2] estimated the effective composite compliance matrix as the weighted volume average of the compliance matrices of constituent phases. Hill [3] has proven that for isotropic constituent phases and composites, Voigt estimation and Reuss estimation, respectively, provide the upper and lower bounds for the effective bulk and shear moduli of composites.

Although the upper and lower bounds of the effective Young's modulus of composites are not given explicitly in previous works, two special composite layouts (serial connection as shown in Fig. 1 and parallel connection as shown in Fig. 2) are always used in many textbooks and literatures (e.g., Ref. [4]) to investigate these bounds. The constituent phases are isotropic elastic with Young's moduli E_A and E_B, volume fractions Φ_A and Φ_B, respectively.

By neglecting the Poisson effects, the serial and parallel layouts are essentially one-dimensional models, and satisfy the isostrain (Voigt) and isostress (Reuss) conditions. Based on Hill's work [3], the bounds for the Young's modulus of the composite $E_{\text{composite}}$ therefore can be given as

$$\tilde{E}_{\text{Reuss}} \leqslant E_{\text{composite}} \leqslant \tilde{E}_{\text{Voigt}}, \tag{1}$$

where

$$\frac{1}{\tilde{E}_{\text{Reuss}}} = \frac{\Phi_A}{E_A} + \frac{\Phi_B}{E_B}, \quad \text{or } \tilde{E}_{\text{Reuss}} = \frac{E_A E_B}{\Phi_A E_B + \Phi_B E_A}, \tag{2}$$

and

$$\tilde{E}_{\text{Voigt}} = \Phi_A E_A + \Phi_B E_B. \tag{3}$$

Here, the overhead tildes in $\tilde{E}_{\text{Reuss}}$ and $\tilde{E}_{\text{Voigt}}$ mean that these effective moduli only approximately satisfy Reuss (isostress) and Voigt (isostrain) conditions due to the neglecting of the Poisson effect.

Based on Eqs. (1)–(3), the following inequalities can be derived and have been widely considered valid for any situation.

Inequality I

$$E_{\text{composite}} \leqslant \tilde{E}_{\text{Voigt}} = \Phi_A E_A + \Phi_B E_B \tag{4}$$

implies that the approximate Voigt estimation $\tilde{E}_{\text{Voigt}}$, i.e., the weighted volume average of the Young's modulus of constituent phases, can be used as the upper bound of the effective Young's modulus of the composite.

Inequality II

$$E_z^{\text{eff}} = E_{\text{composite}}^{\text{serial}} \leqslant E_{\text{composite}}^{\text{parallel}} = E_x^{\text{eff}} \tag{5}$$

implies that the longitudinal (or parallel) stiffness of a layered composite E_x^{eff} is always larger than the transverse (or serial) stiffness E_z^{eff} (see Figs. 1 and 2).

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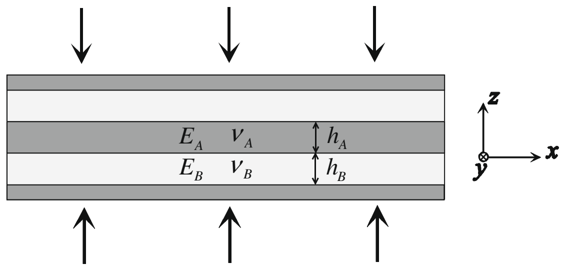

Fig. 1. A schematic diagram of a layer composite under transverse compression.

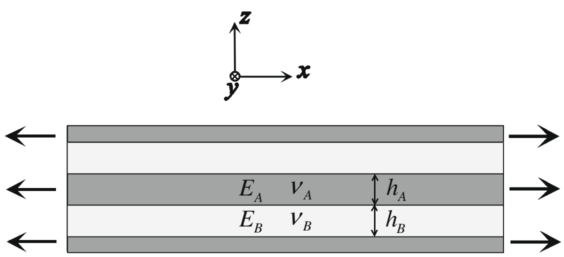

Fig. 2. A schematic diagram of a layer composite under longitudinal tension.

Inequality III

$$E_{\text{composite}} \leqslant \max(E_A, E_B) \tag{6}$$

implies that the composite cannot be stiffer than its stiffest constituent phase.

However, previous studies [5–7] have shown that Poisson effect sometimes can play a very crucial role in the mechanical properties of composites, and should not be ignored. In particular, Liu et al. [5] found that the transverse stiffness of quasi-layered composites is significantly underestimated by the Reuss estimation. Therefore, the purpose of this paper is to investigate the influence of the Poisson effect on the bounds for the Young's modulus of composites and to check the validness of these inequalities Eqs. (1), (4)–(6), which are derived without considering the Poisson effect.

In this paper, a two-phase layered composite as shown in Fig. 1 is used as a specific example to study the Poisson effect. Due to its simple topology, the compliance matrix can be accurately derived without any approximation.

2. The accurate estimation on the stiffness of a two-phase layered composite

Obviously, the layered composite shown in Fig. 1 is a transversally isotropic material, and the x–y plane (see Fig. 1) is the isotropic plane. There are five independent elastic constants for a transversally isotropic material, and the constitutive relation for the composite in terms of the effective compliance matrix is

$$\begin{bmatrix} \bar{\varepsilon}_x \\ \bar{\varepsilon}_y \\ \bar{\varepsilon}_z \\ \bar{\gamma}_{yz} \\ \bar{\gamma}_{zx} \\ \bar{\gamma}_{xy} \end{bmatrix} = \begin{bmatrix} S_{11}^{\text{eff}} & S_{12}^{\text{eff}} & S_{13}^{\text{eff}} & 0 & 0 & 0 \\ & S_{22}^{\text{eff}} & S_{23}^{\text{eff}} & 0 & 0 & 0 \\ & & S_{33}^{\text{eff}} & 0 & 0 & 0 \\ & & & S_{44}^{\text{eff}} & 0 & 0 \\ & (symm) & & & S_{55}^{\text{eff}} & 0 \\ & & & & & S_{66}^{\text{eff}} \end{bmatrix} \begin{bmatrix} \bar{\sigma}_x \\ \bar{\sigma}_y \\ \bar{\sigma}_z \\ \bar{\tau}_{yz} \\ \bar{\tau}_{zx} \\ \bar{\tau}_{xy} \end{bmatrix}, \tag{7}$$

where $[\,\bar{\varepsilon}_x \quad \bar{\varepsilon}_y \quad \bar{\varepsilon}_z \quad \bar{\gamma}_{yz} \quad \bar{\gamma}_{zx} \quad \bar{\gamma}_{xy}\,]^T$ and $[\,\bar{\sigma}_x \quad \bar{\sigma}_y \quad \bar{\sigma}_z \quad \bar{\tau}_{yz} \quad \bar{\tau}_{zx} \quad \bar{\tau}_{xy}\,]^T$ are the strains and stresses of composites, respectively. For transversally isotropic material, the components of the compliance matrix have the following relations due to the symmetry,

$$S_{11}^{\text{eff}} = S_{22}^{\text{eff}}, \quad S_{13}^{\text{eff}} = S_{23}^{\text{eff}}, \quad S_{44}^{\text{eff}} = S_{55}^{\text{eff}}, \quad S_{66}^{\text{eff}} = 2(S_{11}^{\text{eff}} - S_{12}^{\text{eff}}). \tag{8a--d}$$

The superscript "eff" in this paper implies the effective quantities for composites.

To obtain the elastic constants of the composite, we apply three loadings to the representative volume element (RVE) shown in Fig. 1 and then compute the deformation.

2.1. The layered composite under transverse compression (only $\bar{\sigma}_z \neq 0$)

It is easy to know that under the normal compression (see Fig. 1), the stress and the strain in each phase of RVE are uniform, and they should satisfy the following equations:

Constitutive equations:

$$\varepsilon_x^A = \frac{1}{E_A}\left(\sigma_x^A - \nu_A \sigma_y^A - \nu_A \sigma_z^A\right), \tag{9}$$

$$\varepsilon_y^A = \frac{1}{E_A}(\sigma_y^A - \nu_A \sigma_x^A - \nu_A \sigma_z^A), \tag{10}$$

$$\varepsilon_z^A = \frac{1}{E_A}(\sigma_z^A - \nu_A \sigma_x^A - \nu_A \sigma_y^A), \tag{11}$$

$$\varepsilon_x^B = \frac{1}{E_B}(\sigma_x^B - \nu_B \sigma_y^B - \nu_B \sigma_z^B), \tag{12}$$

$$\varepsilon_y^B = \frac{1}{E_B}(\sigma_y^B - \nu_B \sigma_x^B - \nu_B \sigma_z^B), \tag{13}$$

$$\varepsilon_z^B = \frac{1}{E_B}(\sigma_z^B - \nu_B \sigma_x^B - \nu_B \sigma_y^B). \tag{14}$$

Here, σ and ε are stress and strain; E and ν are Young's modulus and Poisson ratio; the superscripts or subscripts "A" and "B" denote phase A and phase B, respectively. It is noted that there is no shear stress and strain in this situation, so they are not included in the equations above.

Equilibrium equations:

$$\bar{\sigma}_x = \frac{h_A \sigma_x^A + h_B \sigma_x^B}{h_A + h_B} = 0, \tag{15}$$

$$\bar{\sigma}_y = \frac{h_A \sigma_y^A + h_B \sigma_y^B}{h_A + h_B} = 0, \tag{16}$$

$$\bar{\sigma}_z = \sigma_z^A = \sigma_z^B. \tag{17}$$

Here, h_A and h_B are the layer thickness of phase A and phase B (see Fig. 1), respectively.

Kinematic equations:

$$\bar{\varepsilon}_x = \varepsilon_x^A = \varepsilon_x^B, \tag{18}$$

$$\bar{\varepsilon}_y = \varepsilon_y^A = \varepsilon_y^B, \tag{19}$$

$$\bar{\varepsilon}_z = \frac{\varepsilon_z^A h_A + \varepsilon_z^B h_B}{h_A + h_B}. \tag{20}$$

By solving Eqs. (9)–(20), we can compute the Young's modulus along z-direction (or transverse direction) as

$$E_z^{\text{eff}} = \frac{1}{S_{33}^{\text{eff}}} = \frac{\bar{\sigma}_z}{\bar{\varepsilon}_z} = \frac{E_A E_B}{\Phi_A E_B + \Phi_B E_A - \frac{2\Phi_A \Phi_B (\nu_A E_B - \nu_B E_A)^2}{(1-\nu_A)\Phi_B E_B + (1-\nu_B)\Phi_A E_A}}, \tag{21}$$

where $\Phi_A = h_A/(h_A + h_B)$ and $\Phi_B = h_B/(h_A + h_B)$ are the volume fractions of phase A and phase B, respectively. It should be pointed out that all derivations in this paper have been checked by the mathematical software Maple. Another compliance component can also be obtained from this loading situation

$$S_{13}^{\text{eff}} = \frac{\bar{\varepsilon}_x}{\bar{\sigma}_z} = \frac{\Phi_A \nu_A + \Phi_B \nu_B - \nu_A \nu_B}{\Phi_A \nu_B E_A + \Phi_B \nu_A E_B - \Phi_A E_A - \Phi_B E_B}. \tag{22}$$

Remark 1. The assumptions used in the Reuss (isostress) approximation are essentially $\sigma_x^A = \sigma_x^B = 0$ and $\sigma_y^A = \sigma_y^B = 0$, which are too