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A fifth-order approximation to gravity-capillary interfacial waves of infinite depth

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ABSTRACT

Two-dimensional periodic gravity-capillary waves at the interface between two unbounded fluids with different density are analyzed. The lighter fluid is above the interface. The perturbation method is used to obtain solutions to the fifth order for interface profile, velocity potential and oscillation frequency. The solutions have been carefully controlled by other solutions (third-order surface gravity-capillary solutions, third-order interface gravity waves and fifth-order surface gravity waves). These solutions can be used to describe the qualitative nature of small-amplitude traveling waves and provide initial guesses for numerical solutions to the full Euler system. The results highlight the significant influence on wave profile and wave frequency. In addition, this study extends the Wilton singularity to interfacial waves.

1. Introduction

A great number of studies has been conducted on two-dimensional progressive gravity-capillary surface waves. The interested reader is referred to the excellent literature review article of Dias and Kharif (Dias & Kharif [1]). A similar study can also be carried out for two-dimensional progressive gravity-capillary waves, propagating at the interface between two fluids of different densities. Hunt [2] applied the Levi-Civita's method for progressive and standing interface waves and obtained formulae of the wave profile and phase velocity up to the third order. He showed that the existence of upper layer reduces the propagation velocity and the amplitude of the higher harmonics in the wave profile. Unfortunately, this paper is incorrect, since the boundary conditions are incorrectly applied. Tsuji & Nagata [3] used a perturbation expansion in wave amplitude to the fifth order, for interfacial gravity waves of infinite depths. Their results suggested that the maximum value of the wave steepness may be limited by shear instability at the interface rather than the breaking condition at the interface. Holyer [4], on the other hand, calculated the maximum steepness for interfacial waves numerically up to 31st order. He showed that surface waves do not break at the crest if we consider the density of the air. Increasing further the

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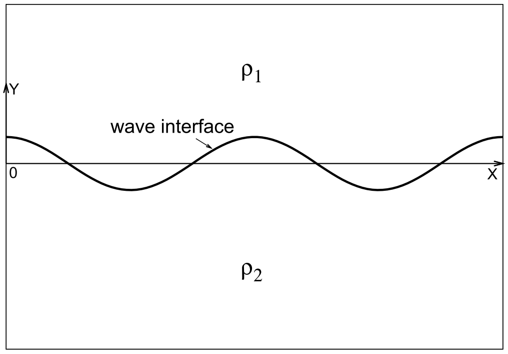

Fig. 1. The configuration of the problem.

density ratio yields an increase in both the maximum amplitude allowed by the waves and the breaking location. Boussinesq waves correspond to the limiting case of equal densities, at which the breaking location occurs at 1/4 wavelength. Saffman & Yuen [5] considered finite-amplitude interfacial gravity waves between two unbounded fluids in relative motion and identified two different factors that limit the existence of steady gravity wave solutions. Părău & Dias [6] computed periodic waves of permanent form that propagate into a fluid system with free surface boundary conditions using Fourier series expansions. Hill [7] studied analytically the weakly nonlinear cubic interactions between surface waves and interfacial waves. He obtained a set of third-order equations describing the interactions between surface waves and interfacial waves. His study revealed the importance of cubic interactions. Yuan, Li & Cheng [8] established a diagram that demarcates the validity ranges for interfacial wave theories in a two-layer system. The proposed diagram is an extension of Le Méhauté's plot for free surface waves. Liu & Hwung [9] extended the work of Saffman & Yuen [5] to the case of two finite depths. The solutions are derived using the perturbation method. Recently, Grigor'ev, Shiryaeva & Sukhanov [10] investigated the Kelvin–Helmholtz instability of gravity-capillary waves. The lower fluid is assumed inviscid while the upper fluid is a dielectric with translational motion parallel to the interface. They showed that when the density ratio is greater than one, the resonances are absent. Liu, Hwung & Yang [11] considered the case of a two-fluid system with free surface. Second-order solutions were obtained, using the perturbation method. The aim of this study is to derive an analytical solution for interfacial progressive waves of two unbounded fluids. The fifth-order perturbation solution is presented. Moreover, the kinematic properties such as wave profile and frequency are investigated by considering the effects of capillary number and density ratio.

2. Problem formulation

Periodic gravity-capillary waves at the interface between two unbounded fluids are considered. The wave is assumed to move from left to right without change of form along an interface under the influence of gravity and surface tension. The fluids are supposed to be incompressible and inviscid, and the motion is assumed to be irrotational. The properties of the upper fluid are denoted by (1), and those of the lower fluid by (2). We present here the properties of finite-amplitude periodic waves with wavelength λ, which propagate steadily without change of shape with speed C. It is convenient to change the framework in order to reduce the wave propagation to rest by moving with the wave. The flow is then independent of time, and is sketched in Fig. 1. The two fluids are assumed to be stable and stratified, so $\rho_1 < \rho_2$. Rectangular coordinates (x, y) are chosen such that the x-axis is horizontal and the y-axis is directed vertically upwards.

Since both fluids are incompressible and the motion in each fluid is irrotational, we can define the velocity potentials that satisfy Laplace's equation

$$\triangle\phi_1 = 0, \qquad \triangle\phi_2 = 0 \tag{1}$$

subject to the following boundary conditions

$$\phi_1 = 0 \quad \text{for} \quad y \longrightarrow \infty \tag{2}$$

$$\phi_2 = 0 \quad \text{for} \quad y \longrightarrow -\infty \tag{3}$$

$$\eta_t + \phi_{ix}\eta_x - \phi_{iy} = 0 \quad \text{at} \quad y = \eta \quad i = (1, 2) \tag{4}$$

$$\rho_2\left[\phi_{2t} + \frac{1}{2}\left(\phi_{2x}^2 + \phi_{2y}^2\right)\right] - \rho_1\left[\phi_{1t} + \frac{1}{2}\left(\phi_{1x}^2 + \phi_{1y}^2\right)\right] + g\left(\rho_2 - \rho_1\right)\eta - \sigma\frac{\eta_{xx}}{\left(1 + \eta_x^2\right)^{3/2}} = 0 \quad \text{at} \quad y = \eta \tag{5}$$

where g denotes the acceleration due to gravity and σ the surface tension coefficient.

For waves of permanent form, we have

$$\eta_t = -C\eta_x \quad \text{and} \quad \phi_{it} = -C\phi_{ix} \quad i = 1, 2 \tag{6}$$

Substituting (6) into (4) and (5), it follows that