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On scattering frequencies in homogenization problems. Critical cases

Sur les fréquences de diffusion dans les problèmes d'homogénéisation. Cas critiques

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ABSTRACT

We consider a two-dimensional boundary value problems for the Helmholtz equation with Dirichlet and Neumann boundary conditions on a set of arcs. This set is obtained from a closed curve by cutting out small holes situated close to each other and having a locally periodic structure. We construct the asymptotics of the scattering frequencies (poles of the analytic continuation of solutions) with small imaginary parts, which converge to the square roots of multiple eigenvalues of limit problems.

RÉSUMÉ

On considère des problèmes aux limites bidimensionnels pour l'équation de Helmholtz avec conditions aux limites de Dirichlet ou de Neumann sur un ensemble d'arcs. Cet ensemble est obtenu à partir d'une courbe fermée en découpant des petits trous situés près les uns des autres, avec une structure localement périodique. Nous construisons le développement asymptotique des fréquences de diffusion (pôles du prolongement analytique des solutions) avec des parties imaginaires petites, qui convergent vers les racines carrées des valeurs propres multiples du problème limite.

1. Introduction

It is known that the scattering of E-polarized electromagnetic field on the ideally conducting cylindrical surface, the cross-section of which is the curve Γ_δ, and the vibrations of a membrane fixed on Γ_δ, are described by the solution to the following Dirichlet boundary value problem in $\Omega_\delta = \mathbb{R}^2 \backslash \overline{\Gamma_\delta}$:

$$(\Delta + k^2)u_\delta = f, \quad x \in \Omega_\delta, \quad u_\delta = 0, \quad x \in \Gamma_\delta, \quad \frac{\partial u_\delta}{\partial r} - \mathrm{i}ku_\delta = o(r^{-1/2}), \quad r \to \infty \tag{1}$$

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where $x = (x_1, x_2)$, $r = |x|$, k is a positive number. In its turn, the following Neumann boundary value problem

$$(\Delta + k^2)u_\delta = f, \quad x \in \Omega_\delta, \quad \frac{\partial u_\delta}{\partial \mathbf{n}} = 0, \quad x \in \Gamma_\delta, \quad \frac{\partial u_\delta}{\partial r} - \mathrm{i}ku_\delta = o(r^{-1/2}), \quad r \to \infty \tag{2}$$

where $\mathbf{n}$ is the normal to Γ_δ, describes an H-polarized electromagnetic field on the ideally conducting cylindrical surface, the cross-section of which is the curve Γ_δ, and the vibrations of a membrane not fixed on the cut Γ_δ.

Hereinafter, we assume that $\delta \geqslant 0$, $\Gamma_0 \in C^\infty$ is a boundary of a bounded simply-connected domain Ω and, for $\delta = \varepsilon > 0$, the curve Γ_ε is obtained from Γ_0 by cutting out a great number of openings of small diameter located close to each other. Namely, let ω be a unit circle with its center at the origin. Suppose that $\gamma_0 = \partial\omega$, $N \gg 1$ is an integer number, $\varepsilon = 2N^{-1}$. For the boundary value problem (1), we assume that $\gamma_\varepsilon = \{(r,\theta) : r = 1,\ \varepsilon(-a(\varepsilon) + m\pi) < \theta < \varepsilon(a(\varepsilon) + m\pi),\ m = 0, 1, \dots, N-1\}$, where θ is the polar angle and $0 < a(\varepsilon) < \frac{\pi}{2}$, and for the boundary value problem (2), we assume that $\gamma_\varepsilon = \{(r,\theta) : r = 1,\ \varepsilon(a(\varepsilon) + m\pi) < \theta < \varepsilon(\pi(m+1) - a(\varepsilon)),\ m = 1, \dots, N\}$. We denote $\Omega = \mathcal{P}(\omega)$, $\Gamma_\delta = \mathcal{P}(\gamma_\delta)$, $\mathcal{P}$ is a diffeomorphism $\mathbb{R}^2$ into $\mathbb{R}^2$ and assume that $\mathbf{n}$ is the outer normal to Ω. For $\delta = \varepsilon$, we will call the problems (1) and (2) the perturbed problems. Since $\Omega_0 = \Omega \cup (\mathbb{R}^2 \backslash \overline{\Omega})$, it follows that for $\delta = 0$ the problem (1) decomposes into two Dirichlet problems, in Ω and in $\mathbb{R}^2 \backslash \overline{\Omega}$, and the problem (2) decomposes into two Neumann problems, in Ω and in $\mathbb{R}^2 \backslash \overline{\Omega}$. We will call them the internal limit and the external limit problems, respectively.

In the following, we assume that

$$\varepsilon \ln a(\varepsilon) \underset{\varepsilon \to 0}{\to} 0 \tag{3}$$

for the boundary value problem (1) and

$$a(\varepsilon) = \exp\left(-\frac{1}{\varepsilon A(\varepsilon)}\right), \quad A(\varepsilon) > 0, \quad \lim_{\varepsilon \to 0} A(\varepsilon) = 0 \quad \varepsilon \to 0 \tag{4}$$

for the boundary value problem (2).

For these cases, it is known [1] (see also [2]) that if $\lambda = k^2$ is not the eigenvalue of the limit internal problem, then a solution to the perturbed problem converges to solutions to the limit problems in Ω and $\mathbb{R}^2 \backslash \overline{\Omega}$. Assume that $k_0 > 0$ and $\lambda_0 = k_0^2$ is an eigenvalue of the limit internal problem. In [3] it is shown that the analytic continuation (with respect to k) of the solution to the perturbed problem has a pole τ^ε with small imaginary part (a scattering frequency), converging to k_0 as $\varepsilon \to 0$. This pole lies in the lower complex semi-plane $\operatorname{Im} k < 0$, and the residue of the analytic continuation of the solution is a solution to the boundary value problem

$$\left(\Delta + (\tau^\varepsilon)^2\right)\psi^\varepsilon = 0, \quad x \in \Omega_\varepsilon, \qquad \psi^\varepsilon = 0, \quad x \in \Gamma_\varepsilon \tag{5}$$

for (1) and a solution to the boundary value problem

$$\left(\Delta + (\tau^\varepsilon)^2\right)\psi^\varepsilon = 0, \quad x \in \Omega_\varepsilon, \qquad \frac{\partial \psi^\varepsilon}{\partial \mathbf{n}} = 0, \quad x \in \Gamma_\varepsilon \tag{6}$$

for (2). Let us emphasize that for fixed ε the function ψ^ε increases exponentially as $r \to \infty$. We will call it a quasi-eigenfunction.

In the case when $\lambda_0 = k_0^2$ is a simple eigenvalue of the limit internal problems, the leading terms of the asymptotics of pole converging to k_0 and of the associated quasi-eigenfunction were constructed in [4,5]. In the present paper, we constructed the leading terms of the asymptotics of poles converging to k_0 and of the associated quasi-eigenfunctions in cases when the multiplicity of an eigenvalue $\lambda_0 = k_0^2$ of the limit internal problems is equal to $n \geqslant 2$.

2. Asymptotics of quasi-eigenelements for the boundary value problem (1)

In this section we assume that the condition (3) holds. Let λ_0 be an eigenvalue of multiplicity $n \geqslant 2$ of the limit interior boundary value problem for (1), and let $\psi_0^{(l)}$ $(l = 1, 2, \dots, n)$ be the corresponding orthonormal eigenfunctions in $L_2(\Omega)$, i.e.,

$$-\Delta\psi_0^{(l)} = \lambda_0\psi_0^{(l)}, \quad x \in \Omega, \qquad \psi_0^{(l)} = 0, \quad x \in \Gamma_0$$

$$\int_\Omega \left(\psi_0^{(l)}(x)\right)^2 \mathrm{d}x = 1, \qquad \int_\Omega \psi_0^{(l)}(x)\psi_0^{(p)}(x)\,\mathrm{d}x = 0, \qquad l \neq p, \quad l, p = \overline{1,n}$$

The leading terms of the asymptotics for the poles $\tau^{\varepsilon,l}$ and associated quasi-eigenfunctions $\psi^{\varepsilon,l}$ outside a neighborhood of Γ_0 are constructed in the form

$$\tau^{\varepsilon,l} = k_0 + \varepsilon\tau_1^{(l)} + \varepsilon^2\tau_2^{(l)} + \dots \tag{7}$$

$$\psi^{\varepsilon,l}(x) = \psi_0^{(l)}(x) + \varepsilon\psi_1^{(l)}(x) + \varepsilon^2\psi_2^{(l)}(x) + \dots, \quad x \in \Omega \tag{8}$$

$$\psi^{\varepsilon,l}(x) = \varepsilon\Psi_1^{(l)}(x; \tau^{\varepsilon,l}) + \varepsilon^2\Psi_2^{(l)}(x; \tau^{\varepsilon,l}) + \dots, \quad x \in \mathbb{R}^2 \backslash \overline{\Omega} \tag{9}$$