C. R. Mecanique 344 (2016) 190–194

Contents lists available at ScienceDirect

Comptes Rendus Mecanique

www.sciencedirect.com

Elastic and piezoelectric waveguides may have infinite number of gaps in their spectra

Les guides d'ondes élastiques et piézoélectriques peuvent avoir un nombre infini de lacunes dans leur spectre

Sergei A. Nazarov [a,b,c], Jari Taskinen [d,*]

[a] *Saint-Petersburg State University, Universitetskaya nab., 7–9, St. Petersburg, 199034, Russia*
[b] *Peter the Great Saint-Petersburg State Polytechnical University, Polytechnicheskaya ul., 29, St. Petersburg, 195251, Russia*
[c] *Institute of Problems of Mechanical Engineering RAS, V.O., Bolshoj pr., 61, St. Petersburg, 199178, Russia*
[d] *University of Helsinki, Department of Mathematics and Statistics, P.O. Box 68, 00014 Helsinki, Finland*

ARTICLE INFO

Article history:
Received 4 December 2015
Accepted 18 December 2015
Available online 28 January 2016

Keywords:
Waveguide
Elasticity
Piezoelectricity
Essential spectrum
Spectral gap
Singular Weyl sequence
Parametrix

Mots-clés :
Guide d'ondes
Élasticité
Piézoélectricité
Spectre essentiel
Troux spectraux
Suite de Weyl
Paramétrix

ABSTRACT

We consider elastic and piezoelectric waveguides composed from identical beads threaded periodically along a spoke converging at infinity. We show that the essential spectrum constitutes a non-negative monotone unbounded sequence and thus has infinitely many spectral gaps.

RÉSUMÉ

Nous considérons des guides d'ondes élastiques et piézoélectriques réalisés à partir de perles identiques agencées de façon périodique le long d'un rayon convergeant à l'infini. Nous montrons que le spectre essentiel est une suite croissante positive non bornée. Cela prouve l'existence d'un nombre infini de trous spectraux.

1. Elastic waveguide

Let $\varpi \subset \{x = (y, z) \in \mathbb{R}^2 \times \mathbb{R} : z = x_3 \in (0, 1)\}$ be a domain with Lipschitz boundary $\partial\varpi$ and compact closure $\overline{\varpi} = \varpi \cup \partial\varpi$, where the surface $\partial\varpi$ has two planar parts $\gamma_p = \mathbb{B}^2_R \times \{p\}$ with $p = 0, 1$ and $\mathbb{B}^2_R = \{y : |y| < R\}$. We also introduce a Lipschitz domain $\omega \subset \mathbb{B}^2_{R/2}$ and an infinitesimal sequence $\{\alpha_j\}_{j\in\mathbb{Z}} \subset (0, 1)$, where $\mathbb{Z} = \{0, \pm1, \pm2, \dots\}$, as well as the sets

* Corresponding author.
E-mail addresses: s.nazarov@spbu.ru, srgnazarov@yahoo.co.uk (S.A. Nazarov), jari.taskinen@helsinki.fi (J. Taskinen).

http://dx.doi.org/10.1016/j.crme.2015.12.004

$$\varpi_j = \{x : (y, z - j) \in \varpi\}, \quad \omega_j = \{x : \alpha_j^{-1} y \in \omega, z = j\} \quad \text{for } j \in \mathbb{Z} \tag{1}$$

The waveguide Π (Fig. 1a),

$$\Pi = \bigcup_{j\in\mathbb{Z}} (\varpi_j \cup \omega_j) \tag{2}$$

consists of identical beads threaded periodically along a thin spoke, which converges at infinity. In other words, the neighboring beads ϖ_{j-1} and ϖ_j are connected through the aperture ω_j.

Let waveguide (2) be filled with a homogeneous elastic material. The variational formulation of the elasticity problem on time-harmonic oscillations with a frequency $\kappa > 0$ reads as

$$(AD(\nabla)u, D(\nabla)v)_\Pi = \lambda(u, v)_\Pi + (f, v)_\Pi \quad \forall v \in H^1(\Pi) \tag{3}$$

Here, $\lambda = \rho\kappa^2$ and $\rho > 0$ are the spectral parameter and the material density, $\nabla = \text{grad}$, f denotes mass forces and the Voigt–Mandel notation is in use so that $u = (u_1, u_2, u_3)^\top \in \mathbb{R}^3$ and $\varepsilon(u) = D(\nabla)u \in \mathbb{R}^6$ are the displacement and strain columns,

$$D(\nabla)^\top = \begin{pmatrix} \partial_1 & 0 & 0 & 0 & \alpha\partial_3 & \alpha\partial_2 \\ 0 & \partial_2 & 0 & \alpha\partial_3 & 0 & \alpha\partial_1 \\ 0 & 0 & \partial_3 & \alpha\partial_2 & \alpha\partial_1 & 0 \end{pmatrix}, \ \alpha = \frac{1}{\sqrt{2}}, \ \partial_j = \frac{\partial}{\partial x_j}, \ \nabla = \begin{pmatrix} \partial_1 \\ \partial_2 \\ \partial_3 \end{pmatrix}$$

and $\top$ stands for transposition. Furthermore, A denotes a symmetric and positive definite 6×6-matrix of constant elastic moduli in the Hooke's law $\sigma(u) = AD(\nabla)u$, where $\sigma_{jk}(u)$ are Cartesian components of the stress tensor composing the stress column

$$\sigma(u) = (\sigma_{11}(u), \sigma_{22}(u), \sigma_{33}(u), \sqrt{2}\sigma_{23}(u), \sqrt{2}\sigma_{31}(u), \sqrt{2}\sigma_{12}(u))^\top$$

Finally, $(\cdot,\cdot)_\Pi$ is the natural inner product in the Lebesque space $L^2(\Pi)$ and $H^1(\Pi)$ is the Sobolev space, either scalar or vector. The problem (3) realizes as the continuous mapping $B - \lambda : H^1(\Pi) \to H^1_0(\Pi)^*$, where $H^1_0(\Pi)^*$ is the dual space.

In principle, problem (3) should be supplied with appropriate radiation conditions, see [1] for a scalar problem and [2] for elasticity. However, since we will only consider the inhomogeneous ($f \neq 0$) problem in the regularity field, it is not necessary to formulate them.

2. Motivation

The structure (2) of waveguide Π, which consists of the periodic family of identical cells (1) connected through converging apertures, comes from the previous works of the authors [3–6] and their attempts to prove or disprove the existence of infinite number of spectral gaps for periodic elastic and piezoelectric waveguides. This question is related to the classical Bethe–Sommerfeld conjecture on a finite number of spectral gaps for any periodic waveguide in $\mathbb{R}^d$, $d > 1$. The conjecture has been solved for some scalar problems only, mainly for the stationary Schrödinger equation, see [7–10] and [11] for an introduction to the topic. The (periodic) waveguide Π^ε in papers [3–6] was obtained as the union of the cells (1) and the thin cylinder $\Omega^\varepsilon = \{(y, z) : \varepsilon^{-1} y \in \omega, z \in \mathbb{R}\}$ where the domain $\omega \subset \mathbb{R}^{d-1}$ is as above, but $\varepsilon > 0$ is a small parameter. In other words, the cells are connected through small but *fixed* apertures $\omega_j^\varepsilon = \{(y, z) : \varepsilon^{-1} y \in \omega, z = j\}$. By constructing asymptotics of eigenvalues of a model problem in ϖ obtained from (3) in Π^ε by the Gelfand transform, it was proven that, for any $N \in \mathbb{N} = \{1, 2, 3, \dots\}$, there exists $\varepsilon_N > 0$ such that the spectrum of the problem in Π^ε with $\varepsilon \in (0, \varepsilon_N]$ has at least N opened spectral gaps. However, such asymptotic analysis does not seem to suffice for opening infinitely many gaps, because $\varepsilon_N \to +0$ as $N \to +\infty$. On the other hand, the waveguide (2) is not periodic because diameters $O(\alpha_j)$ of the apertures ω_j connecting the identical beads ϖ_j and ϖ_{j+1} decay as $j \to \pm\infty$. Hence, Theorem 1 does not solve the Bethe–Sommerfeld conjecture.

3. Spectrum

According to the Korn inequality in ϖ, see, e.g., [12], the left-hand side of the integral identity (3) constitutes a closed positive Hermitian form in $H^1(\Pi)$. Hence, problem (3) is associated with a positive self-adjoint unbounded operator $\mathcal{A}$ in $L^2(\Pi)$ with a domain $\mathcal{D}(\mathcal{A}) \subset H^1(\Pi)$, see [13, Ch. 10]. Its spectrum $\wp$ belongs to the closed positive real semi-axis $\overline{\mathbb{R}_+}$ and

$$\wp = \wp_{\text{di}} \cup \wp_{\text{es}}, \quad \wp_{\text{di}} \cap \wp_{\text{es}} = \varnothing \tag{4}$$

where $\wp_{di}$ and $\wp_{\text{es}}$ are the discrete and essential spectra, respectively. To describe the latter component, we mention that the model problem of the elasticity theory in the bounded Lipschitz domain ϖ

$$(AD(\nabla)U, D(\nabla)V)_\varpi = \Lambda(U, V)_\varpi \quad \forall V \in H^1(\varpi) \tag{5}$$

possesses the eigenvalue sequence