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Elasticity of fractal materials using the continuum model with non-integer dimensional space

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A R T I C L E I N F O

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A B S T R A C T

Using a generalization of vector calculus for space with non-integer dimension, we consider elastic properties of fractal materials. Fractal materials are described by continuum models with non-integer dimensional space. A generalization of elasticity equations for non-integer dimensional space, and its solutions for the equilibrium case of fractal materials are suggested. Elasticity problems for fractal hollow ball and cylindrical fractal elastic pipe with inside and outside pressures, for rotating cylindrical fractal pipe, for gradient elasticity and thermoelasticity of fractal materials are solved.

1. Introduction

Fractals are measurable metric sets with non-integer Hausdorff dimensions [1,2]. The main characteristic of a fractal set is the non-integer Hausdorff dimension that should be observed on all scales. The Hausdorff dimension is a local property, i.e. this dimension characterize (measure) property of a set of distributed points in the limit of a vanishing diameter, which is used to cover subset of the points. By definition the Hausdorff dimension requires the knowledge of the diameter of the covering sets to vanish. In general, real materials have a characteristic smallest length scale R_0 such as the radius of a particle such as an atom or a molecule. In fractal materials, the fractal structure cannot be observed on all scales, but only on those for which $R > R_0$, where R_0 is the characteristic scale of the particles. For real materials, a non-integer mass dimension can be used instead of the Hausdorff dimension. The mass dimension described how the mass of a medium region scales with the size of this region, where we assume an unchanged density. For many cases, we have an asymptotic relation between the mass $M(W)$ of a ball region W of material, and the radius R of this ball. The mass of fractal material satisfies a power-law relation $M(W) \sim R^D$. The parameter D is called the non-integer mass dimension of a fractal material. This parameter does not depend on the shape of the region W, or on whether the packing of sphere of radius R_0 is a close packing, a random packing or a porous packing with a uniform distribution of holes. Therefore a fractal material can be considered as a medium with non-integer mass dimension. Although the non-integer dimension does not reflect completely the geometrical and dynamical properties of the fractal materials, it nevertheless permits to draw a number of important conclusions about the behavior of the materials. It allows us to use effective models that take into account non-integer dimensions.

We can distinguish the following approaches to formulate models of fractal materials.

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1) An approach based on methods of "analysis on fractals" [3–8] can be considered as the most rigorous approach to describe fractal materials. Unfortunately, a possibility of application of the "analysis on fractals" to solve real problems of fractal material is now very limited due to the weak development of this area of mathematics.

2) To describe a fractal material, we can apply a special continuum models suggested in [9–12] and then developed in the works [13–15,73]. These models can be called the fractional-integral continuum models. In this approach, we use integrations of non-integer orders, and two different notions such as the density of states and the distribution function [15]. The order of the fractional integrals is equal to the mass dimension of fractal materials. The kernels of these integrals are defined by the power-law type of the density of states.

3) Fractional derivatives of non-integer orders are used to describe some properties of fractal materials. This approach has been suggested in papers [16–18], where the so-called local fractional derivatives are used, and then developed in the works [19–25]. These models can be called the fractional-differential models.

4) Fractal materials can be described by using the theory of integration and differentiation for non-integer dimensional spaces [26–28]. Fractal materials are described as continuum in non-integer dimensional spaces. The dimensions of the spaces are equal to the mass dimensions of fractal materials.

Unfortunately, there are not enough differential equations that are solved for various problems for fractal materials in the framework of the fractional-differential model and by methods of "analysis on fractals".

The fractional-integral continuum models are used to solve differential equations for various problems of elasticity of fractal materials [29–33,14], and of thermoelasticity of fractal materials [34,35].

Continuum models with non-integer dimensional spaces are not currently used to describe the elasticity of fractal materials. In this paper, we consider an approach based on the non-integer dimensional space to describe the elasticity of isotropic fractal materials. The main difference of the continuum models with non-integer dimensional spaces and fractional-integral continuum models suggested in [9–12,15] may be reduced to the following: (a) the arbitrariness in the choice of the numerical factor in the density of states is fixed by the equation of the volume of non-integer dimensional ball region. (b) In the fractional-integral continuum models, the differentiations are integer orders, whereas the integrations are non-integer orders. In the continuum models with non-integer dimensional spaces, the integrations and differentiations are defined for the spaces with non-integer dimensions. The power law $M \sim R^D$ can be naturally derived by using the integrations in a non-integer dimensional space [26], whose dimension is equal to the mass dimension of the fractal material.

A vector calculus for non-integer dimensional space proposed in this paper allows us to use continuum models, which are based on non-integer dimensional space, to describe fractal materials. This is due to the fact that although the non-integer dimension does not reflect all geometrical and dynamical properties of the fractal materials, it nevertheless allows us to get important results about the behavior of fractal materials. Therefore continuum models with non-integer dimensional spaces can describe a wide class of fractal materials.

Integration over non-integer dimensional spaces is actively used in the theory of critical phenomena and phase transitions in statistical physics [36,37], and in the dimensional regularization of ultraviolet divergences in quantum field theory [38,39,26]. The axioms for integrations in a non-integer dimensional space are proposed in [40,27] and this type of integration is considered in the book by Collins [26] for rotationally covariant functions. In the paper [27], a mathematical basis of integration on a non-integer dimensional space is given, and a generalization of the Laplace operator for non-integer dimensional spaces is suggested. Using a product measure approach, the Stillinger's methods [27] have been generalized by Palmer and Stavrinou [28] for multiple variables case with different degrees of confinement in orthogonal directions. The scalar Laplace operators suggested by Stillinger in [27] and by Palmer, Stavrinou in [28] for non-integer dimensional spaces, have successfully been used for effective descriptions in physics and mechanics. The Stillinger's form of the Laplacian for the Schrödinger equation in a non-integer dimensional space is used by He [41–43] to describe a measure of the anisotropy and confinement by the effective non-integer dimensions. Quantum mechanical models with non-integer (fractional) dimensional space have been discussed in [27,28,44–48] and [49–52]. Recent progress in non-integer dimensional space approach also includes a description of the fractional diffusion processes in a non-integer dimensional space in [53], and the electromagnetic fields in a non-integer dimensional space in [54–56] and [57–60].

Unfortunately, in the articles [27,28], only second-order differential operators for scalar fields in the form of the scalar Laplacian for the non-integer dimensional space are proposed. A generalization of the vector Laplacian [61] for the non-integer dimensional space is not suggested. The first-order operators such as gradient, divergence, curl operators, and vector Laplacian are not considered in [27,28] also. In the work [62], the gradient, divergence, and curl operators are suggested only as approximations of the square of the Laplace operator. Consideration only the scalar Laplacian in non-integer dimensional space approach greatly restricts us in application of continuum models with non-integer dimensional space for fractal materials and material. For example, we cannot use the Stillinger's form of Laplacian for the displacement vector field $\mathbf{u}(\mathbf{r}, t)$ in the theory of elasticity and thermoelasticity of fractal materials, for the velocity vector field $\mathbf{v}(\mathbf{r}, t)$ in hydrodynamics of fractal fluids, for electric and magnetic vector fields in electrodynamics of fractal media in the framework of the non-integer dimensional space approach.

In this paper, we define the first- and second-order differential vector operations such as gradient, divergence, the scalar and vector Laplace operators for a non-integer dimensional space. In order to derive the vector differential operators in a non-integer dimensional space, we use the method of analytic continuation in dimension. For the sake of simplification, we consider rotationally covariant scalar and vector functions that are independent of angles. This allows us to reduce differential equations in non-integer dimensional space to ordinary differential equations with respect to r. The proposed operators