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Stochastic modeling of a class of stored energy functions for incompressible hyperelastic materials with uncertainties

Sur une classe de potentiels élastiques stochastiques pour les matériaux hyperélastiques incompressibles isotropes

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ABSTRACT

In this Note, we address the construction of a class of stochastic Ogden's stored energy functions associated with incompressible hyperelastic materials. The methodology relies on the maximum entropy principle, which is formulated under constraints arising in part from existence theorems in nonlinear elasticity. More specifically, constraints related to both polyconvexity and consistency with linearized elasticity are considered and potentially coupled with a constraint on the mean function. Two parametric probabilistic models are thus derived for the isotropic case and rely in part on a conditioning with respect to the random shear modulus. Monte Carlo simulations involving classical (*e.g.*, Neo-Hookean or Mooney–Rivlin) stored energy functions are then performed in order to illustrate some capabilities of the probabilistic models. An inverse calibration involving experimental results is finally presented.

RÉSUMÉ

Dans cette Note, on s'intéresse à la construction d'une classe de modèles stochastiques pour des matériaux hyperélastiques incompressibles. La méthodologie de construction repose sur le principe du maximum d'entropie, formulé à partir de contraintes induites par les théorèmes d'existence en élasticité non linéaire. Plus précisément, des contraintes associées à la polyconvexité et à la cohérence avec l'élasticité linéarisée sont introduites, et éventuellement couplées avec une contrainte relative à la fonction moyenne. Deux modèles probabilistes paramétriques pour les densités d'énergie considérées sont par suite proposés dans le cas isotrope, et reposent notamment sur un conditionnement vis-à-vis du module de cisaillement aléatoire. Des simulations numériques de Monte Carlo pour des potentiels classiques (*e.g.*, Néo-Hookéen ou Mooney–Rivlin) sont ensuite conduites afin d'illustrer les capacités du modèle. Une identification inverse basée sur des résultats expérimentaux est enfin présentée.

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1. Introduction

In this work, we address the construction of parametric probabilistic representations for stored energy functions defining incompressible hyperelastic materials. Such models are dedicated to predictive modeling in nonlinear elasticity, where engineered or biological materials can exhibit uncertainties (at some scale of interest) that are worth taking into account. This random behavior may arise from, *e.g.*, batch-to-batch variability or processing defects for manufactured composites, or from intrinsic variability in the case of complex heterogeneous materials (see [1] and the references therein for a discussion regarding experimental results in biomechanics, for instance). Unlike the linear case, where the modeling and propagation of uncertainties gave rise to an extensive literature (both in applied mathematics and computational mechanics; see the references below), the nonlinear case has surprisingly received little attention – at least, from a modeling standpoint. Uncertainty propagation from microscale to macroscale was analytically addressed in [2], where bounds on effective properties are expressed, by means of Hashin–Shtrikman bounds, in terms of the fluctuation terms. Computational multiscale frameworks based on a microscale description were further proposed in [3,4] and rely on the combination between interpolation schemes (in the space of macroscopic deformations) and polynomial chaos expansions [5]. The construction of relevant parametric probabilistic representations for stored energy functions exhibiting some uncertainties therefore remains an intricate and open question. In this paper, we propose a very first contribution to this field in the framework of Information Theory, and restrict the derivations to the isotropic case for the sake of readability. The aim is to develop a methodology for the derivation of relevant probabilistic models for stochastic stored energy functions, thanks to the principle of maximum entropy. In order to ensure mathematical consistency, the latter is formulated under constraints related to existence theorems in nonlinear elasticity and coherence at small strains. These constraints can be subsequently supplemented with an additional one associated with the mean function, if need be. The paper is organized as follows. For completeness, a brief review of hyperelasticity is first presented in Section 2. The construction of a class of stochastic stored energy functions for incompressible hyperelastic materials is then addressed in Section 3. Monte Carlo simulations and an inverse identification based on experimental data are finally presented in Section 4 in order to illustrate the model capabilities.

Notation Throughout this paper, use will be made of the following matrix sets:

(i) $\mathbb{M}_d(\mathbb{R})$ the set of real $(d \times d)$ matrices;
(ii) $\mathbb{L}_d(\mathbb{R})$ the set of real $(d \times d)$ matrices with an unitary determinant.

Deterministic (resp. stochastic) scalar-valued random variables are denoted α or a (resp. $\mathbb{\alpha}$ or A). Similarly, deterministic (resp. stochastic) vectors are denoted by $\boldsymbol{a}$ (resp. $\boldsymbol{A}$).

2. Framework for deterministic hyperelasticity

Let $\Omega \subset \mathbb{R}^3$ be a bounded open connected set with a sufficiently regular boundary, and denote by $\overline{\Omega}$ its closure. It is assumed that Ω is occupied by a homogeneous incompressible isotropic hyperelastic material characterized by a stored energy function $\widehat{w} : \mathbb{L}_3(\mathbb{R}) \to \mathbb{R}$ such that [6–9]:

$$[\widehat{T}([F])] = \frac{\partial \widehat{w}([F])}{\partial [F]} - \tilde{h}[F]^{-T}, \quad \forall [F] \in \mathbb{L}_3(\mathbb{R}) \tag{1}$$

where $[\widehat{T}] : \mathbb{L}_3(\mathbb{R}) \to \mathbb{M}_3(\mathbb{R})$ is the response function associated with the first Piola–Kirchhoff tensor $[T] : \overline{\Omega} \to \mathbb{M}_3(\mathbb{R})$, $[F]$ is the deformation gradient and $\tilde{h}$ is a Lagrange multiplier (which is typically interpreted as an hydrostatic pressure) enforcing the incompressibility condition. In addition to isotropy, the stored energy function is classically assumed to satisfy frame-invariance, so that according to representation theorems, there exists a function w such that:

$$\widehat{w}([F]) = w(\upsilon_1([F]),\ \upsilon_2([F]),\ \upsilon_3([F])) \tag{2}$$

where $\{\upsilon_j([F])\}_{j=1}^3$ are the eigenvalues of $[F]$. Such a class of strain energy functions was extensively studied for natural rubbers and proposed, on the basis of phenomenological concerns, by Ogden [10,11]. More specifically, the following algebraic form was postulated [10]:

$$\widehat{w}([F]) = \sum_{i=1}^{m} \alpha_i \Phi_{\gamma_i}([F]) + \sum_{j=1}^{n} \beta_j \Upsilon_{\delta_j}([F]), \quad \forall\, [F] \in \mathbb{L}_3(\mathbb{R}) \tag{3}$$

where $\{\alpha_i, \gamma_i\}_{i=1}^m$ and $\{\beta_j, \delta_j\}_{j=1}^n$ are sets of model parameters. In Eq. (3), $\Phi : \mathbb{L}_3(\mathbb{R}) \to \mathbb{R}$ and $\Upsilon : \mathbb{L}_3(\mathbb{R}) \to \mathbb{R}$ are the functions defined as

$$\Phi_{\gamma_i}([F]) = \upsilon_1([F])^{\gamma_i} + \upsilon_2([F])^{\gamma_i} + \upsilon_3([F])^{\gamma_i} - 3 \tag{4}$$