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High-frequency cell vibrations and spatial skin effect in thick cascade junction with heavy concentrated masses

Vibrations cellulaires de haute fréquence et effet spatial « peau » dans une jonction cascade épaisse avec des masses lourdes concentrées

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ABSTRACT

We study the asymptotic behavior of eigenvalues and eigenfunctions of the Laplacian in a 2D thick cascade junction with heavy concentrated masses. We present two-term asymptotic approximations, as $\varepsilon \to 0$, for the eigenelements in the case of "slightly heavy", "moderate heavy", and "super heavy" concentrated masses. Asymptotics of high-frequency cell-vibrations are found as well.

RÉSUMÉ

Nous étudions le comportement asymptotique des valeurs et fonctions propres du laplacien dans une jonction cascade épaisse bidimensionnelle, avec des masses lourdes concentrées. Si $\varepsilon \to 0$, nous présentons des approximations asymptotiques en deux termes pour les éléments propres dans les cas des masses concentrées « peu lourdes », « modérément lourdes » et « super-lourdes ». L'analyse asymptotique pour les vibrations à haute fréquence cellulaire est aussi trouvée.

1. Introduction

In this paper, we present the results obtained for a spectral problem with heavy concentrated masses in a *thick cascade junction*. Vibrating systems with a concentration of masses on a small set of diameter $\mathcal{O}(\varepsilon)$ have been studied for a long time. It was experimentally established that such concentration leads to the big reduction of the main frequencies and to the large localization of vibrations near concentrated masses. New impulse in this research was given by E. Sánchez-Palencia in

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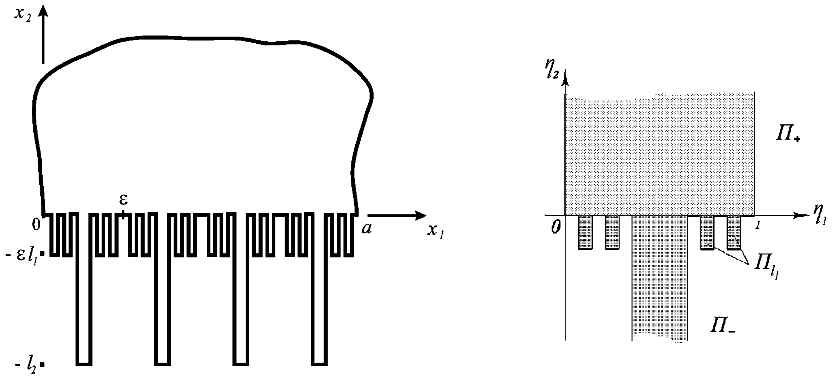

Fig. 1. Thick cascade junction Ω_ε (left) and cell of periodicity Π (right).

the paper [1], in which the effect of local vibrations was mathematically described. After this paper, many articles appeared. The reader can find a widely presented bibliography on spectral problems with concentrated masses in [2–11].

1.1. Statement of the problem

Let a, b_1, b_2, h_1, h_2 be positive numbers, $0 < b_1 < b_2 < \frac{1}{2}$, $0 < b_1 - \frac{h_1}{2}$, $b_1 + \frac{h_1}{2} < b_2 - \frac{h_1}{2}$, $b_2 + \frac{h_1}{2} < \frac{1}{2} - \frac{h_2}{2}$.

Let Ω_0 be a bounded domain in $\mathbb{R}^2$ with the Lipschitz boundary $\partial\Omega_0$ and $\Omega_0 \subset \{x := (x_1, x_2) \in \mathbb{R}^2\colon\ x_2 > 0\}$. Let $\partial\Omega_0$ contain the segment $I_0 = \{x\colon\ x_1 \in [0, a],\ x_2 = 0\}$. We also assume that there exists a positive number δ_0 such that $\Omega_0 \cap \{x\colon\ 0 < x_2 < \delta_0\} = \{x\colon\ x_1 \in (0, a),\ x_2 \in (0, \delta_0)\}$.

We divide $[0, a]$ into segments $[\varepsilon j, \varepsilon(j+1)]$, $j = 0, \dots, N-1$, $N \in \mathbb{N}$; $\varepsilon = a/N$ is a small discrete parameter.

A model *thick cascade junction* Ω_ε (see Fig. 1) consists of the junction's body Ω_0 and a large number of thin rods $G_j^{(1)}(d_k, \varepsilon) = \{x \in \mathbb{R}^2\colon\ |x_1 - \varepsilon(j + d_k)| < \frac{\varepsilon h_1}{2},\ x_2 \in (-\varepsilon l_1, 0]\}$, $k = 1, \dots, 4$, $G_j^{(2)}(\varepsilon) = \{x \in \mathbb{R}^2\colon\ |x_1 - \varepsilon(j + \frac{1}{2})| < \frac{\varepsilon h_2}{2},\ x_2 \in (-l_2, 0]\}$, $j = 0, \dots, N-1$, where $d_1 = b_1$, $d_2 = b_2$, $d_3 = 1 - b_2$, $d_4 = 1 - b_1$, that is $\Omega_\varepsilon = \Omega_0 \cup G_\varepsilon^{(1)} \cup G_\varepsilon^{(2)}$, where $G_\varepsilon^{(1)} = \bigcup_{j=0}^{N-1}(\bigcup_{k=1}^{4} G_j^{(1)}(d_k, \varepsilon))$, $G_\varepsilon^{(2)} = \bigcup_{j=0}^{N-1} G_j^{(2)}(\varepsilon)$.

In Ω_ε we consider the following spectral problem

$$\begin{cases} -\Delta_x u(\varepsilon, x) = \lambda(\varepsilon)\rho_\varepsilon(x)u(\varepsilon, x), & x \in \Omega_\varepsilon; \qquad u(\varepsilon, x) = 0, \qquad x \in \Gamma_1 \\ -\partial_\nu u(\varepsilon, x) = 0, & x \in \partial\Omega_\varepsilon \setminus \Gamma_1; \qquad [u]|_{x_2=0} = [\partial_{x_2} u]|_{x_2=0} = 0, \quad x_1 \in Q_\varepsilon \end{cases} \tag{1}$$

Here $\partial_\nu = \partial/\partial\nu$ is the outward normal derivative; the brackets denote the jump of the enclosed quantities; Γ_1 is a curve on $\partial\Omega_0$, located in $\{x\colon\ x_2 > \delta_0\}$; the density $\rho_\varepsilon(x) = 1$, $x \in \Omega_0 \cup G_\varepsilon^{(2)}$ and $\rho_\varepsilon(x) = \varepsilon^{-\alpha}$, $x \in G_\varepsilon^{(1)}$; the parameter $\alpha > 0$; $Q_\varepsilon = Q_\varepsilon^{(1)} \cup Q_\varepsilon^{(2)}$, $Q_\varepsilon^{(i)} = G_\varepsilon^{(i)} \cap \{x\colon\ x_2 = 0\}$, $i = 1, 2$.

Obviously, that for each fixed value of ε there is a sequence of eigenvalues:

$$0 < \lambda_1(\varepsilon) < \lambda_2(\varepsilon) \leqslant \dots \leqslant \lambda_n(\varepsilon) \leqslant \dots \to +\infty \quad \text{as } n \to \infty \tag{2}$$

of problem (1). The corresponding eigenfunctions $\{u_n(\varepsilon, \cdot)\}_{n\in\mathbb{N}}$, which belong to $\mathcal{H}_\varepsilon$, can be orthonormalized as follows $(u_n, u_k)_{L_2(\Omega_0 \cup G_\varepsilon^{(2)})} + \varepsilon^{-\alpha}(u_n, u_k)_{L_2(G_\varepsilon^{(1)})} = \delta_{n,k}$, $\{n, k\} \in \mathbb{N}$. Here and below $\delta_{n,k}$ is the Kronecker delta, $\mathcal{H}_\varepsilon$ is the Sobolev space $\{u \in H^1(\Omega_\varepsilon)\colon\ u|_{\Gamma_1} = 0$ in sense of the trace$\}$ with the scalar product $(u, v)_{\mathcal{H}_\varepsilon} := \int_{\Omega_\varepsilon} \nabla u \cdot \nabla v \, dx\ \forall u, v \in \mathcal{H}_\varepsilon$.

Our aim is to study the asymptotic behavior of the eigenvalues $\{\lambda_n(\varepsilon)\}_{n\in\mathbb{N}}$ and the eigenfunctions $\{u_n(\varepsilon, \cdot)\}_{n\in\mathbb{N}}$ as $\varepsilon \to 0$ if $\alpha > 1$, to find other limiting points of the spectrum of problem (1), and to describe the corresponding eigenoscillations.

We establish five qualitatively different cases in the asymptotic behavior of eigenvalues and eigenfunctions of problem (1) as $\varepsilon \to 0$, namely the case of "light" concentrated masses ($\alpha \in (0, 1)$), "intermediate" concentrated masses ($\alpha = 1$), and "heavy" concentrated masses ($\alpha \in (1, +\infty)$) that we divide into "slightly heavy" concentrated masses ($\alpha \in (1, 2)$), "moderately heavy" concentrated masses ($\alpha = 2$), and "super heavy" concentrated masses ($\alpha > 2$).

In the cases of "light" and "intermediate" concentrated masses, the perturbation of domain plays the leading role in the asymptotic behavior of the eigenelements. These cases were completely studied in [2], where we proved the low- and high-frequency convergences of the spectrum of problem (1) as $\varepsilon \to 0$, we constructed and justified the leading terms of the asymptotics both for the eigenfunctions and the eigenvalues; in addition, as in the paper [9], we found *pseudovibrations* in problem (1), having a rapidly oscillating character, and in which different rods of the junction vibrate individually, i.e., each rod has its own frequency.