A Riemannian approach to strain measures in nonlinear elasticity

Une approche riemannienne vers des mesures des deformation en élasticité non linéaire

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A B S T R A C T

The isotropic Hencky strain energy appears naturally as a distance measure of the deformation gradient to the set SO(n) of rigid rotations in the canonical left-invariant Riemannian metric on the general linear group GL(n). Objectivity requires the Riemannian metric to be left-GL(n)-invariant, isotropy requires the Riemannian metric to be right-O(n)-invariant. The latter two conditions are only satisfied for a three-parameter family of Riemannian metrics on the tangent space of GL(n). Surprisingly, the final result is basically independent of the chosen parameters.

In deriving the result, geodesics on GL(n) have to be parameterized and a novel minimization problem, involving the matrix logarithm for non-symmetric arguments, has to be solved.

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R É S U M É

L'énergie isotrope de Hencky est une mesure naturelle de la distance du gradient de déformation à l'ensemble des rotations rigides SO(n) dans la métrique riemanienne canonique du groupe linéaire GL(n). Le principe d'indifférence matérielle exige que la métrique soit GL(n)-invariante à gauche, et l'isotropie implique son invariance à droite par O(n). Ces deux conditions sont uniquement satisfaites par une famille à trois paramètres de métriques riemanniennes sur l'espace tangent à GL(n). On note cependant que le résultat final se révèle, en essence, indépendant des paramètres choisis. Pour obtenir ce résultat, on effectue une paramétrisation des géodésiques de GL(n) et l'on résout un problème de minimisation qui fait intervenir le logarithme de matrices non symétriques.

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1. Introduction

For the deformation gradient $F = \nabla\varphi \in \mathrm{GL}^+(n)$ let $U = \sqrt{F^T F}$ be the symmetric right Biot-stretch tensor. We show that the isotropic Hencky strain energy, which was introduced by H. Hencky in 1928 [1] and is defined on the logarithmic strain tensor $\log U$ via

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$$W(F) = \mu \| \operatorname{dev} \log U \|^2 + \frac{\kappa}{2} \big[\operatorname{tr}(\log U) \big]^2 = \mu \| \operatorname{dev} \log U \|^2 + \frac{\kappa}{2} (\log \det F)^2 \tag{1}$$

measures the geodesic distance of F to the group of rotations $\mathrm{SO}(n)$ where $\mathrm{GL}(n)$ is viewed as a Riemannian manifold endowed with a left-invariant metric that is also right $\mathrm{O}(n)$-invariant (isotropic), and where the coefficients $\mu, \kappa > 0$ correspond to the shear modulus and the bulk modulus, respectively. Furthermore, the results provide yet another characterization of the polar decomposition $F = RU$, $R \in \mathrm{SO}(n)$, $U \in \mathrm{PSym}(n)$, since U also provides the minimal Euclidean distance to $\mathrm{SO}(n)$, i.e. [2],

$$\operatorname{dist}^2_{\mathrm{euclid}}\big(F, \mathrm{SO}(n)\big) := \min_{Q \in \mathrm{SO}(n)} \operatorname{dist}^2_{\mathrm{euclid}}(F, Q) = \min_{Q \in \mathrm{SO}(n)} \| F - Q \|^2 = \| F - R \|^2 = \| U - \mathbb{1} \|^2 \tag{2}$$

where the Euclidean distance $\operatorname{dist}^2_{\mathrm{euclid}}(X, Y) := \| X - Y \|^2$ is the length of the line segment joining X and Y in $\mathbb{R}^{n^2}$, $\mathbb{1} \in \mathrm{GL}^+(n)$ is the identity and $\| X \| = \sqrt{\operatorname{tr}(X^T X)}$ denotes the Frobenius matrix norm here and henceforth. For both the Euclidean and the geodesic distance, the orthogonal factor $R = \operatorname{polar}(F)$ in the polar decomposition of F is the nearest rotation to F.

2. Strain tensors in linear and nonlinear elasticity

We consider an elastic body which in a reference configuration occupies the bounded domain $\Omega \subset \mathbb{R}^3$. Deformations of the body are prescribed by mappings $\varphi : \Omega \to \mathbb{R}^3$, where $\varphi(x)$ denotes the deformed position of the material point $x \in \Omega$. Central to elasticity theory is the notion of strain, which is a tensor depending on the deformation such that vanishing strain implies that the body Ω has been moved rigidly in space. Various such tensors exist, e.g. the *Green strain tensor* $\frac{1}{2}(U^2 - \mathbb{1})$, the *generalized Green strain tensor* $\frac{1}{m}(U^m - \mathbb{1})$, where m is a nonzero integer, and the *Hencky* (or *logarithmic*) *strain tensor* $\log U$.

In linearized elasticity, one considers $\varphi(x) = x + u(x)$, where $u : \Omega \subset \mathbb{R}^3 \to \mathbb{R}^3$ is the displacement. The classical linearized strain tensor is $\varepsilon = \operatorname{sym} \nabla u$. It appears through a matrix-nearness problem in the Euclidean distance

$$\operatorname{dist}^2_{\mathrm{euclid}}\big(\nabla u, \mathfrak{so}(3)\big) := \min_{W \in \mathfrak{so}(3)} \| \nabla u - W \|^2 = \| \operatorname{sym} \nabla u \|^2 \tag{3}$$

where $\mathfrak{so}(3)$ denotes the set of all skew symmetric matrices in $\mathbb{R}^{3 \times 3}$. Indeed, $\operatorname{sym} \nabla u$ qualifies as a linearized strain tensor: if $\operatorname{dist}^2_{\mathrm{euclid}}(\nabla u, \mathfrak{so}(3)) = 0$ then $u(x) = \widehat{W}.x + \widehat{b}$ is a linearized rigid movement. This is the case since

$$\operatorname{dist}^2_{\mathrm{euclid}}\big(\nabla u(x), \mathfrak{so}(3)\big) = 0 \quad \Rightarrow \quad \nabla u(x) = W(x) \in \mathfrak{so}(3) \tag{4}$$

and $0 = \operatorname{Curl} \nabla u(x) = \operatorname{Curl} W(x)$ implies that $W(x)$ is constant, see [3]. In nonlinear elasticity theory, one assumes that $\nabla \varphi \in \mathrm{GL}^+(3)$ (no self-interpenetration of matter) and may consider the matrix nearness problem

$$\operatorname{dist}^2_{\mathrm{euclid}}\big(\nabla \varphi, \mathrm{SO}(3)\big) := \min_{Q \in \mathrm{SO}(3)} \| \nabla \varphi - Q \|^2 = \min_{Q \in \mathrm{SO}(3)} \| Q^T \nabla \varphi - \mathbb{1} \|^2 = \| \sqrt{\nabla \varphi^T \nabla \varphi} - \mathbb{1} \|^2, \tag{5}$$

where the last equality is due to (2). Indeed, the Biot strain tensor $\sqrt{\nabla \varphi^T \nabla \varphi} - \mathbb{1}$ qualifies as a nonlinear strain tensor: if $\operatorname{dist}^2_{\mathrm{euclid}}(\nabla \varphi, \mathrm{SO}(3)) = 0$ then $\varphi(x) = \widehat{Q}.x + \widehat{b}$ is a rigid movement. This is the case since

$$\operatorname{dist}^2_{\mathrm{euclid}}\big(\nabla \varphi, \mathrm{SO}(3)\big) = 0 \quad \Rightarrow \quad \nabla \varphi(x) = Q(x) \in \mathrm{SO}(3) \tag{6}$$

and $0 = \operatorname{Curl} \nabla \varphi(x) = \operatorname{Curl} Q(x)$ implies that $Q(x)$ is constant, see [3].

In geometrically nonlinear, physically linear isotropic elasticity, the formulation of a boundary value problem of place may now be based on minimizing the quadratic Biot strain energy

$$\mathcal{E}(\varphi) = \int_{\Omega} \mu \big\| \operatorname{dev}[\sqrt{\nabla \varphi^T \nabla \varphi} - \mathbb{1}] \big\|^2 + \frac{\kappa}{2} \big(\operatorname{tr} \sqrt{\nabla \varphi^T \nabla \varphi} - \mathbb{1} \big)^2 \mathrm{dx}, \quad \varphi|_{\Gamma_D} = \varphi_0, \tag{7}$$

where $\mu, \kappa > 0$ are the shear modulus and bulk modulus, respectively.

However, since the Euclidean distance in (5) is an arbitrary choice, novel approaches in nonlinear elasticity theory aim at putting more geometry (i.e. respecting the group structure of the deformation mappings) into the description of the strain a material endures. In our context, it is now natural to consider a strain measure induced by the geodesic distances stemming from choices for the Riemannian structure respecting also the algebraic group structure of $\mathrm{GL}^+(n)$, which we introduce next.

3. Left invariant Riemannian metrics on GL(n)

Viewing $\mathrm{GL}(n)$ as a Riemannian manifold endowed with a left invariant metric:

$$g_H : T_H \mathrm{GL}(n) \times T_H \mathrm{GL}(n) \to \mathbb{R} : \quad g_H(X, Y) = \langle H^{-1} X, H^{-1} Y \rangle, \quad H \in \mathrm{GL}(n), \tag{8}$$