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Theoretical and numerical approaches for Vlasov–Maxwell equations

# Case studies in space charge and plasma acceleration of charged beams

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ABSTRACT

Plasma acceleration with electron or proton driver beams is a challenging opportunity for high-energy physics. An energy doubling experiment with electron drivers was successfully performed at SLAC and a key experiment AWAKE with proton drivers is on schedule at CERN. Simulations play an important role in choosing the best experimental conditions and in interpreting the results. The Vlasov equation is the theoretical tool to describe the interaction of a driver particle beam or a driver laser pulse with a plasma. Collective effects, such as tune shift and mismatch instabilities, appear in high intensity standard accelerators and are described by the Poisson–Vlasov equation. In the paper, we review the Vlasov equation in the electrostatic and fully electromagnetic cases. The general framework of variational principles is used to derive the equation, the local form of the balance equations and related conservation laws. In the electrostatic case, we remind the analytic Kapchinskij–Vladimirskij (K–V) model and we propose an extension of the adiabatic theory for Hamiltonian systems, which ensures stability for perturbation of size $\epsilon$ on times of order $1/\epsilon$. The variational framework is used to derive the Maxwell–Vlasov equations and related conservation laws and to briefly sketch the particle-in-cell (PIC) approximation schemes. Finally, the proton-driven acceleration is examined in the linear and quasi-linear regime. A PIC simulation with the code `ALaDyn` developed at Bologna University is presented to illustrate the longitudinal and transverse fields evolution which allow a witness electron bunch to be accelerated with a gradient of a few GeV/m. We also present some remarks on future perspectives.



## 1. Introduction

Particle accelerators are key instruments for the research in the domain of high-energy particle physics. To explore to even further accuracy the Standard Model, higher energies are required, thus pushing the technology of particle accelerators to its limits. The challenge is being tackled in two different domains, namely the design of high-field magnets, required to control the motion of high-energy charged particles, and that of high-gradient radio-frequency (RF) systems, aimed at

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accelerating the charged particles to the energies required for probing the intimate nature of matter and forces. Collective effects, such as space charge, are relevant in the early accelerating stages and limit the intensity reachable in final ring. Their control allows one to increase the final luminosity, a key parameter for high-energy experiments. In the domain of accelerating systems, the standard approach consists in generating longitudinal electric fields using metallic RF cavities, either normal or superconducting. Technological limits arise when trying to reach accelerating gradients of the order of 100 MV/m, which would have a direct impact on the dimensions of future accelerators. In order to keep under control the size of the new instruments, gradients of the order of GV/m would be needed and a plasma could be the natural solution, as it can sustain very large accelerating fields [1–3]. It is worth recalling the efforts devoted in the last few decades to experimental verification of plasma acceleration, which reached very encouraging results [4,5]. It is also important to stress the two approaches that can be used, namely laser wakefield acceleration (LWFA) [1], whenever a laser is driving the acceleration, or plasma wakefield acceleration (PWFA) [2,3], whenever a bunch is used to drive the whole process. The fundamental principle behind both approaches is that a plasma acts as an energy transformer, i.e., transferring the energy from the driver bunch to the witness bunch. As proton bunches at the energy frontier, i.e., in the multi-TeV range, are already available, it has been proposed the use of protons, instead of electrons, as driver beams open up the possibility of generating extremely high-energy lepton beams via PWFA [6]. Furthermore, the use of protons to drive the acceleration process would be a première, never tried before.

In this paper we review the Vlasov theory and the related numerical schemes. We first examine the electrostatic case that allows one to model the space-charge effects in a beam, which are relevant at low–medium energies. Since space charge mainly affects the transverse motion, one- and two-dimensional models are adequate to describe the basic dynamical features. The analytic Kapchinskij–Vladimirskij solution [7,8], applicable to a linear lattice, and the adiabatic theory stability results, applicable to a flat beam, allow us to understand many properties of the beam dynamics such as the mismatch oscillations. However, numerical solutions for two- or three-dimensional models provided by the particle-in-cell method (PIC) are needed to define stationary solutions to the Vlasov equation and to study their stability, in particular when multipolar magnetic components are present in the accelerator lattice.

In the more general framework of laser plasma or beam plasma acceleration, the theoretical description is provided by Maxwell–Vlasov equations as long as collisional effects can be neglected. In this case, analytical results are available only for some one-dimensional models and one has essentially to rely on the numerical solutions provided by the PIC schemes. In spite of the remarkable computational power presently available, the simulation of the acceleration on long plasma channels along thousands plasma lengths is still a challenge and reduced schemes, where the fastest scales are neglected, must be adopted. The control of accuracy is important, especially for reduced schemes, and the presence of conserved variables is helpful. In this paper, we present a short review of the theoretical framework for the electromagnetic case applicable to plasma acceleration. As an example of numerical computations, we finally consider the acceleration of electrons driven by proton bunches like the AWAKE experiment on schedule at CERN.

The plan of the paper is as following: in Section 2, we introduce the Poisson Vlasov theory via the action principle, which is suitable to obtain the equations of motion and the conservation laws. In Section 3, we consider the space charge problem in a ring after recalling the Kapchinskij–Vladimirskij model and the procedure to develop numerical schemes. In Section 4, the stability problem for a flat beam is discussed using the Hamiltonian adiabatic theory and it is proved that the size of an instability does not overcome $\epsilon \sim \|\partial f/\partial t\|$ for a time interval at least $1/\epsilon$. In Section 5 we consider the Maxwell–Vlasov theory starting from Low's Lagrangian [9]. The way to obtain the local energy balance for the electromagnetic field and the charged fluid is sketched. In Section 6 we outline the procedure to obtain the PIC schemes with an interpolating basis of linear splines and sampling the distribution functions of the charged fluid with numerical particles. In Section 7 we briefly review the proposal of AWAKE experiment, discussing the acceleration of electrons driven by a proton beam in a plasma in the linear and quasi-linear regimes. The results of a simulation obtained with the code `ALaDyn` developed at Bologna University are presented to illustrate the accelerating longitudinal field created in the plasma. A summary and comments on future perspectives conclude the presentation.

## 2. The Poisson–Vlasov equation

We consider a system of $N$ particles of charge $e$ and mass $m$ interacting via Coulomb repulsion and subject to an external electrostatic field with potential $V^{(\mathrm{ex})}$. Denoting with $H_{\mathrm{tot}}$ the total Hamiltonian, with $M = Nm$ and $Q = Ne$ the total mass and charge, we write

$$\frac{H_{\mathrm{tot}}}{M} = \frac{1}{N}\sum_{k=1}^{N} H_k(\mathbf{x}_k, \mathbf{p}_k)$$

$$H_k(\mathbf{x}_k, \mathbf{p}_k) = \frac{\mathbf{p}_k^2}{2} + \frac{Q\,e}{m}\frac{1}{2N}\sum_{j,(j\neq k)}\frac{1}{r_{jk}} + \frac{e}{m}V^{(\mathrm{ex})}(\mathbf{x}_k)$$

where $\mathbf{x}, \mathbf{p} \in \mathbb{R}^3$. Each $H_k$ is the Hamiltonian for a single particle, with an external potential $V_k(\mathbf{x})$ generated by all the particles $i \neq k$. In the limit $N \to \infty$, the $N$ body system is replaced by a continuum, whose phase space distribution $f(\mathbf{x}, \mathbf{p}, t)$