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A robust and well-balanced numerical model for solving the two-layer shallow water equations over uneven topography

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ABSTRACT

A robust and well-balanced numerical model is developed for solving the two-layer shallow water equations based on the approximate Riemann solver in the framework of finite-volume methods. The HLL (Harten, Lax, and van Leer) solver is employed to calculate the numerical fluxes. The numerical balance between the flux gradient and the source terms is achieved by using a balance-reformulation method. To obtain exactly the lake-at-rest solutions as the water depth is chosen as the conserved variable for the continuity equations, a modified HLL flux formulation is proposed for mass flux calculations. Several numerical tests used to validate the performance of the developed numerical model. The results show that the developed model is accurate, well balanced, and that it predicts no oscillations around large gradients.

1. Introduction

In this study, we consider a system with two superposed shallow flow layers. The fluids are assumed to be immiscible and of different but constant densities. Mathematically resolving of the two-layer shallow water equations (abbreviated as 2LSWE hereafter) is useful and of great importance for studying stratified flow motions, e.g., wave–mud interactions [1,2], thermally or salinity-driven exchange flow motions [3], internal wave behaviors in the continental shelf [4], and chaotic mixing of particles in layered flows [5].

Existing numerical schemes for solving the 2LSWE rely on the numerical techniques for solving the one-layer shallow water equation (1LSWE hereafter). In solving the 1LSWE, various tough issues were encountered, e.g., numerical oscillation or instability around shocks or discontinuities, generation of spurious flow over uneven topography caused by the numerical imbalance between the flux gradient and the source terms. The numerical techniques for resolving these difficulties related to the 1LSWE have been significantly advanced in recent years [6–8]. Compared with the situation in solving the 1LSWE, solving the 2LSWE is more complicated, since the coupling between the layers in the 2LSWE leads to a non-conservative product term. Moreover, it is non-trivial to devise well-balanced numerical schemes for the 2LSWE.

The non-conservative product term in the 2LSWE accounts for the momentum exchange between the layers. This term involves partial derivatives of the unknown physical variables that has no definition around shocks or discontinuities. Rigor-

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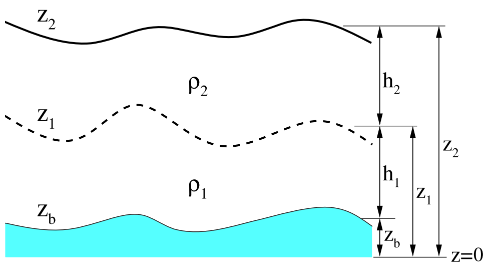

Fig. 1. (Color online.) Definition sketch of a two-layer shallow water system.

ous theory for mathematically resolving the system with non-conservative terms has not been perfected yet. By choosing a specially designed coordinate system, Kurganov and Petrova [9] rewrote the 2LSWE system to make the effect of the nonconservative product term as small as possible. This scheme was verified to be robust when the fluctuation in the upper-layer surface is small. Dumbser et al. [10], and Castro-Díaz et al. [11] developed path-conservative numerical schemes for solving the 2LSWE based on the idea that the non-conservative product term can be interpreted as a Borel measure [12]. As pointed out by Abgrall and Karni [13], the path-conservative schemes may fail in obtaining reasonable solutions; for instance, may capture shocks incorrectly. Spinewine et al. [14] proposed an approach to make the non-conservative product term vanish by rearranging the governing equations.

The well-balanced property (e.g., the C-Property) is another objective to be achieved in developing numerical schemes for solving the 2LSWE involving source terms. The concept of the well-balanced property is originally proposed for solving the 1LSWE; it says that a numerical scheme should achieve a balance between the flux gradient and the source terms under steady stationary flow conditions [15]. A violation of this balance could trigger spurious flows and predict incorrect wave propagation speeds [16,17]. Various methods are proposed in the literature to tackle this imbalance problem in solving the 1LSWE. Several methods are verified to be robust and effective for solving the 1LSWE, e.g., the surface-gradient method [18], the hydraulic-reconstruction approach [19], and the surface-gradient-splitting approach [20,21]. However, when directly applying these approaches to solve the 2LSWE, some are found to be ineffective. For instance, as reported by Lee et al. [22,23], the surface-gradient splitting with a hydraulic reconstruction approach proposed by Liang and Borthwick [21], which exhibits the exactly well-balanced property for solving the 1LSWE, induces high spurious oscillations in the vicinity of discontinuities when used for solving the 2LSWE. The primary reason for explaining the different performances of a same method in solving the two different equation systems, lies in the existence of the non-conservative product term in the 2LSWE. As stated previously, this term is also a tough issue to be addressed.

The aim of this paper is to develop a robust and well-balanced numerical model for solving the 2LSWE with well treatment of the non-conservative product term. Special attention will be made to construct an effective numerical scheme to achieve exact lake-at-rest solutions for solving the 2LSWE.

The rest of the paper is organized as follows. Section 2 presents the numerical method for solving the 2LSWE. The general essentials and techniques for the numerical discretizations are presented. The method to achieve the well-balanced property is detailed. In Section 3, several numerical tests are presented to verify the performances of the numerical model. Finally, conclusions and discussions are given in Section 4.

2. Governing equations and numerical methods

In this paper, attention is focused on 1D flow in a channel with rectangular and constant-width cross sections. In principle, the numerical model developed below can be straightforwardly extended to 2D natural channels with complex geometry. The definition sketch for the 1D two-layer shallow water system is illustrated in Fig. 1. By neglecting the vertical acceleration effect and adopting the hydrostatic assumption, the 1D 2LSWE can be derived from the vertical 2D Reynolds-averaged Navier–Stokes equations. The governing equations for the upper layer are

$$\frac{\partial h_2}{\partial t}+\frac{\partial q_{x2}}{\partial x}=0 \tag{1}$$

$$\frac{\partial q_{x2}}{\partial t}+\frac{\partial\left(u_2 q_{x2}+\frac{1}{2}gh_2^2\right)}{\partial x}=-gh_2\frac{\partial z_1}{\partial x}+\frac{\tau_{sx}-\tau_{wx}}{\rho_2} \tag{2}$$

and, for the lower layer,

$$\frac{\partial h_1}{\partial t}+\frac{\partial q_{x1}}{\partial x}=0 \tag{3}$$

$$\frac{\partial q_{x1}}{\partial t}+\frac{\partial\left(u_1 q_{x1}+\frac{1}{2}gh_1^2\right)}{\partial x}=-gh_1\frac{\partial z_b}{\partial x}+\frac{\tau_{wx}-\tau_{bx}}{\rho_1}-\chi gh_1\frac{\partial h_2}{\partial x} \tag{4}$$