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Stokes–Darcy coupling for periodically curved interfaces

Sur les conditions aux limites entre l'équation de Stokes et de Darcy pour une interface courbée

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ABSTRACT

We investigate the boundary condition between a free fluid and a porous medium, where the interface between the two is given as a periodically curved structure. Using a coordinate transformation, we can employ methods of periodic homogenisation to derive effective boundary conditions for the transformed system. In the porous medium, the fluid velocity is given by Darcy's law with a non-constant permeability matrix. In tangential direction as well as for the pressure, a jump appears. Its magnitudes can be calculated with the help of a generalised boundary layer function. The results can be interpreted as a generalised law of Beavers and Joseph for curved interfaces.

RÉSUMÉ

On considère le comportement d'un fluide libre au-dessus d'un milieu poreux avec une interface courbée périodique. Utilisant une transformation des coordonnées, on peut utiliser des méthodes d'homogénéisation périodique pour la dérivation des conditions aux limites. Le comportement du fluide en milieu poreux est donné par une loi de Darcy avec une matrice de perméabilité non constante. Ensuite, on obtient le comportement du fluide à l'interface. Une discontinuité apparaît pour la pression ainsi que pour la vitesse tangentielle. L'amplitude des discontinuités peut être calculée par une fonction de couche limite généralisée. Ainsi, les résultats donnent une loi généralisée de Beavers et Joseph pour des interfaces courbées.

1. Introduction

The interface condition coupling a free flow with a flow in a porous medium is of great interest in mathematical modelling, groundwater flow or soil chemistry, among others. From a physical point of view, the fluid velocity of an incompressible fluid has to be continuous in normal direction to the interface due to mass conservation. However, other conditions are not so obvious due to the different nature of the governing equations: For the free fluid, the Stokes or Navier–Stokes equation is of second order for the velocity and of first order for the pressure, whereas for the Darcy equation in the porous medium the order of the terms is exchanged. By practical experiments, Beavers and Joseph [1] concluded that a jump in

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the effective velocities appears in tangential direction. Using a statistical approach, this condition was verified by Saffman in [2]. However, some parameters in this approach still need to be determined by experiments.

Starting in 1996, Willie Jäger and Andro Mikelić applied the theory of homogenisation to the problem. They first developed a theory of mathematical boundary layers in [3], using these to rigorously derive Saffman's modification of the jump condition:

$$\sqrt{k^{\varepsilon}}\,(\nabla v_{\mathrm{F}}\nu)\cdot\tau = \alpha v_{\mathrm{F}}\cdot\tau + \mathcal{O}\big(k^{\varepsilon}\big) \tag{1}$$

in [4], where v_{F} denotes the velocity of the free fluid at the interface; $k^{\varepsilon} = k\varepsilon^2$ is the (scalar) permeability of the porous medium (where ε denotes its characteristic length), and ν and τ are the unit normal and unit tangential vector, respectively. The slip-coefficient α can be calculated explicitly. They considered a situation that corresponds to the experimental setup of Beavers and Joseph. Later in [5], Mikelić and Marciniak-Czochra extended the results to an arbitrary body force, which gave an additional pressure jump condition. However, all the results above suffer from one drawback: only a planar boundary in the form of a line or a plane is considered. Therefore, the effect of a possible curvature of the interface is not known. Generalisations of the boundary layers in [3] were developed by Maria Neuss-Radu in [6]. However, applications only treat reaction–diffusion systems without flow, and explicit results can only be obtained in the case of a layered medium, see [7].

In [8] we proposed a new approach to consider the case of a non-flat interface by using a coordinate transformation. In this note — using generalised boundary layer functions developed in [9] — we are able to derive boundary conditions of Beavers and Joseph for the case of a periodically curved non-flat interface.

2. Overview of the geometries

In this section we describe the main geometrical settings that are used throughout this work. Let $L, K, h > 0$. Then $\Omega := (0, L)\times(-K, h)$ is a rectangular domain in $\mathbb{R}^2$ (later corresponding to the reference domain) with parts $\Omega_1 := (0, L)\times(0, h)$ (later the reference free fluid domain), $\Omega_2 := (0, L)\times(-K, 0)$ (the reference porous medium) and $\Sigma = (0, L)\times\{0\}$ (later the reference interface). Let $g \in \mathcal{C}^{\infty}(\mathbb{R})$ be a given function such that $g(y + L) = g(y)$ for all $y \in \mathbb{R}$. We consider g to describe a periodic curved structure in our domain of interest. Define the coordinate transformation:

$$\psi : \Omega \longrightarrow \tilde{\Omega}, \qquad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \longmapsto \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 + g(x_1) \end{pmatrix}$$

such that $\tilde{\Omega} = \psi(\Omega)$, $\tilde{\Omega}_1 := \psi(\Omega_1)$, $\tilde{\Omega}_2 := \psi(\Omega_2)$ and $\tilde{\Sigma} := \psi(\Sigma) = \{(y, g(y)) | y \in (0, L)\}$. We are interested in the behaviour of a fluid flowing through the curved channel $\tilde{\Omega}$, where $\tilde{\Omega}_1$ represents a domain with a free fluid flow, and $\tilde{\Omega}_2$ is a porous medium. We are especially interested in the behaviour of the fluid at the curved boundary $\tilde{\Sigma}$. Let $\tilde{\Omega}_S \Subset \tilde{\Omega}_2$ be a given solid inclusion. We will use a sequence of such inclusions to create a porous medium via homogenisation theory.

To do so, define an ε-periodic geometry in Ω_2 by the use of a reference cell $Y := [0, 1]^2$, containing a connected open set Y_{S} (corresponding to the solid part of the cell). Its boundary ∂Y_{S} is assumed to be of class $\mathcal{C}^{\infty}$ with $\partial Y_{\mathrm{S}} \cap \partial Y = \emptyset$. Let $Y^* := Y \backslash \overline{Y_{\mathrm{S}}}$ be the fluid part of the reference cell.

For given $\varepsilon > 0$ such that $\frac{L}{\varepsilon} \in \mathbb{N}$, let χ be the characteristic function of Y^*, extended by periodicity to the whole $\mathbb{R}^2$. Set $\chi^{\varepsilon}(x) := \chi(\frac{x}{\varepsilon})$ and define the fluid part of the porous medium as $\Omega_2^{\varepsilon} = \{x \in \Omega_2 \mid \chi^{\varepsilon}(x) = 1\}$. The fluid domain is then given by $\Omega^{\varepsilon} = \Omega_1 \cup \Sigma \cup \Omega_2^{\varepsilon}$, and the solid part by $\Omega_{\mathrm{S}} = \Omega_2 \backslash \Omega_2^{\varepsilon}$.

In order to obtain the effective fluid behaviour near Σ, we have to define a number of so-called boundary layer problems. To this end, we introduce the following setting: we consider the domain $[0, 1]\times\mathbb{R}$ subdivided as follows: $Z^+ = [0, 1]\times(0, \infty)$ corresponds to the free fluid region, whereas the union of translated reference cells $Z^- = \bigcup_{k=1}^{\infty}\{Y^* - \binom{0}{k}\}\backslash S$ is considered to be the void space in the porous part. Here $S = [0, 1]\times\{0\}$ denotes the interface between Z^+ and Z^-. Finally, let $Z = Z^+ \cup Z^-$ and $Z_{\mathrm{BL}} = Z^+ \cup S \cup Z^-$ be the fluid domain without and with interface.

3. Fluid behaviour at the interface — main results

For a given body force $\tilde{f} \in L^2(\tilde{\Omega})$, we assume that a mathematical description of the fluid is given by the steady-state Stokes equation with no slip condition on the boundary of the solid inclusion and on the outer walls:

$$\begin{aligned}
&-\mu\Delta_z\tilde{u}(z) + \nabla_z\tilde{p}(z) = \tilde{f}(z) && \text{in } \tilde{\Omega}\backslash\overline{\tilde{\Omega}_{\mathrm{S}}} \\
&\mathrm{div}_z\big(\tilde{u}(z)\big) = 0 && \text{in } \tilde{\Omega}\backslash\overline{\tilde{\Omega}_{\mathrm{S}}} \\
&\tilde{u}(z) = 0 && \text{on } \partial\tilde{\Omega}_{\mathrm{S}} \cup \partial\tilde{\Omega}\backslash\big(\{z_1 = 0\}\cup\{z_1 = L\}\big) \\
&\tilde{u}, \tilde{p} && \text{are } L\text{-periodic in } x_1
\end{aligned}$$

Here $\mu > 0$ denotes the dynamic viscosity. We are looking for a velocity field $\tilde{u} \in H^1(\tilde{\Omega})^2$ and a pressure $\tilde{p} \in L^2(\tilde{\Omega})/\mathbb{R}$. The Stokes equation is an approximation of the full Navier–Stokes equation which is valid for low Reynolds number flows. Using the transformation rules for the differential operators (see [8]), we obtain the following equation for the transformed quantities $u^{\varepsilon}(x) = \tilde{u}(\psi(x))$, $p^{\varepsilon}(x) = \tilde{p}(\psi(x))$ and $f(x) = \tilde{f}(\psi(x))$ in the rectangular domain Ω: